Canonical Decomposition, Realizer, Schnyder Labeling and Orderly Spanning Trees of Plane Graphs

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Convex grid drawing

1: all vertices are put on grid points
2: all edges are drawn as straight line segments
3: all faces are drawn as convex polygons
Convex grid drawing

drawing methods
1. canonical decomposition
2. realizer
3. Schnyder labeling
How to construct a convex grid drawing

Canonical Decomposition [CK97]

convex grid drawing but not outer triangular convex grid drawing

outer triangular convex grid drawing
How to construct a convex grid drawing

Realizer [DTV99]
[Fe01]

outer triangular convex grid drawing
How to construct a convex grid drawing

Schnyder labeling [Sc90, Fe01]

outer triangular convex grid drawing
Question 1

Are there any relations between these concepts?

Canonical Decomposition

Convex Grid Drawing

Realizer

Schnyder labeling
**Known results**

3-connectivity is a **sufficient** condition

Canonical Decomposition

Convex Grid Drawing

Realizer

Schnyder labeling
Known results

**Question 2**

What is the necessary and sufficient condition?

- Canonical Decomposition
  - [BTV99]

- Convex Grid Drawing
  - [Fe01]

- Realizer

- Schnyder labeling
  - [Fe01]
Known results

Canonical Decomposition

Convex Grid Drawing

Orderly Spanning Tree

Realizer

Schnyder labeling

[CLL01]

[BTV99]

[Fe01]
Applications of a canonical decomposition

- a realizer
- a Schnyder labeling
- an orderly spanning tree

- convex grid drawing
- floor-planning
- graph encoding
- 2-visibility drawing

etc.
Our results

**Question 1**
Are there any relations between these concepts?

**Question 2**
What is the necessary and sufficient condition?

Canonical Decomposition

Convex Grid Drawing

Orderly Spanning Tree

Realizer

Schnyder labeling
Our results

**Question 2**

What is the necessary and sufficient condition?

Canonical Decomposition

Convex Grid Drawing

Orderly Spanning Tree

Realizer

Schnyder labeling

these five notions are equivalent with each other
Our results

a necessary and sufficient condition for plane graphs for their existence

Canonical Decomposition
Convex Grid Drawing

Orderly Spanning Tree

these five notions are equivalent with each other

Realizer
Schnyder labeling
Theorem \( G \): plane graph with each degree \( \geq 3 \)

(a) - (f) are equivalent with each other.

(a) \( G \) has a canonical decomposition.

(b) \( G \) has a realizer.

(c) \( G \) has a Schnyder labeling.

(d) \( G \) has an outer triangular convex grid drawing.

(e) \( G \) has an orderly spanning tree.

(f) necessary and sufficient condition

- \( G \) is internally 3-connected.
- \( G \) has no separation pair \( \{u, v\} \) such that both \( u \) and \( v \) are on the same \( P_i \) (1 \( \leq i \leq 3 \)).
(f) Our necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same path $P_i$ ($1 \leq i \leq 3$).
(f) Our necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same path $P_i$ ($1 \leq i \leq 3$).

For any separation pair $\{u, v\}$ of $G$,  
1) both $u$ and $v$ are outer vertices, and  
2) each component of $G - \{u, v\}$ contains an outer vertex.
(f) Our necessary and sufficient condition

- $G$ is internally 3-connected.
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(f) Our necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, \nu\}$ s.t. both $u$ and $\nu$ are on the same path $P_i$ ($1 \leq i \leq 3$).

![Diagram showing necessary and sufficient condition with vertices $u$, $\nu$, and paths $P_1$, $P_2$, $P_3$. Each vertex has a degree of at least 3.](image)
(f) Our necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same path $P_i$ ($1 \leq i \leq 3$).
**Theorem**

\(G\) : plane graph with each degree \(\geq 3\)

(a) - (f) are equivalent with each other.

(a) \(G\) has a **canonical decomposition**.

(b) \(G\) has a **realizer**.

(c) \(G\) has a **Schnyder labeling**.

(d) \(G\) has an **outer triangular convex grid drawing**.

(e) \(G\) has an **orderly spanning tree**.

(f) **necessary and sufficient condition**

- \(G\) is internally 3-connected.
- \(G\) has no separation pair \(\{u, v\}\) such that both \(u\) and \(v\) are on the same \(P_i\) \((1 \leq i \leq 3)\).
Our results

necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$).
\( G \) has an outer triangular convex grid drawing

- \( G \) is internally 3-connected
- \( G \) has no separation pair \( \{u, v\} \) s.t. both \( u \) and \( v \) are on the same \( P_i \) (1 \( \leq \) \( i \) \( \leq \) 3)

If

- \( G \) is not internally 3-connected, or
- \( G \) has a separation pair \( \{u, v\} \) such that both \( u \) and \( v \) are on the same \( P_i \) (1 \( \leq \) \( i \) \( \leq \) 3)

Then \( G \) has no outer triangular convex grid drawing.

These faces cannot be simultaneously drawn as convex polygons.

\( G \) has no convex drawing.
$G$ has an outer triangular convex grid drawing

- $G$ is internally 3-connected
- $G$ has no separation pair $\{u,v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

If
- $G$ is not internally 3-connected, or
- $G$ has a separation pair $\{u,v\}$ such that both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

Then $G$ has no outer triangular convex grid drawing.

This face cannot be drawn as a convex polygon.

$G$ has no outer triangular convex grid drawing.
Our results

necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u,v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$).

Canonical Decomposition

Convex Grid Drawing

Orderly Spanning Tree

Realizer

Schnyder labeling

[BTV99]

[Fe01]
• $G$ is internally 3-connected
• $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

$G$ has a canonical decomposition
- \( G \) is internally 3-connected
- \( G \) has no separation pair \( \{u, v\} \) s.t. both \( u \) and \( v \) are on the same \( P_i \) (1 ≤ \( i \) ≤ 3)

\[ G \] has a canonical decomposition

(cd1) \( V_1 \) consists of all vertices on the inner face containing \( \bullet \), and \( V_h = \{ \circ \} \).

(cd2) Each \( G_k \) (1 ≤ \( k \) ≤ \( h \)) is internally 3-connected.

(cd3) All the vertices in each \( V_k \) (2 ≤ \( k \) ≤ \( h - 1 \)) are outer vertices of \( G_k \), and either (a) or (b).
• $G$ is internally 3-connected
• $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

$G$ has a canonical decomposition

(cd1) $V_1$ consists of all vertices on the inner face containing , and $V_h = \{\}$.

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$G$ has a canonical decomposition

(cd1) $V_1$ consists of all vertices on the inner face containing $\bullet \longrightarrow \bullet$, and $V_h=$\{\).

(cd2) Each $G_k$ (1 $\leq$ $k$ $\leq$ $h$) is internally 3-connected.

(cd3) All the vertices in each $V_k$ (2 $\leq$ $k$ $\leq$ $h$ -1) are outer vertices of $G_k$, and either (a) or (b).
• $G$ is internally 3-connected
• $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i \ (1 \leq i \leq 3)$

$G$ has a canonical decomposition

(cd1) $V_1$ consists of all vertices on the inner face containing blue $\rightarrow$ red, and $V_h=\{\text{blue} \}$.

(cd2) Each $G_k \ (1 \leq k \leq h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k \ (2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).
• $G$ is internally 3-connected
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- $G$ has a canonical decomposition

(c1d) $V_1$ consists of all vertices on the inner face containing $\bullet-\bullet$, and $V_h=\{\bullet\}$.

(c2d) Each $G_k$ ($1 \leq k \leq h$) is internally 3-connected.

(c3d) All the vertices in each $V_k$ ($2 \leq k \leq h-1$) are outer vertices of $G_k$, and either (a) or (b).
- $G$ is internally 3-connected
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(a)

(b)
• $G$ is internally 3-connected
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Our results

necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$).

Canonical Decomposition

Convex Grid Drawing

Realizer

Orderly Spanning Tree

Schnyder labeling
Our results

necessary and sufficient condition

• $G$ is internally 3-connected.
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Canonical Decomposition

Convex Grid Drawing

Orderly Spanning Tree

Realizer

Schnyder labeling

[BTV99] [Fe01] [Fe01]
Orderly Spanning Tree

preorder

parent

smaller

larger

children
Realizer

for each vertex
Our results

necessary and sufficient condition

- $G$ is internally 3-connected.
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Canonical Decomposition

Convex Grid Drawing

Orderly Spanning Tree

Realizer

Schnyder labeling

[Fe01]

[BTV99]
Our results:

necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same path $P_i$ ($1 \leq i \leq 3$).

Canonical Decomposition

Convex Grid Drawing

Orderly Spanning Tree

Realizer

Schnyder labeling
Orderly Spanning Tree

Realizer
Orderly Spanning Tree

Realizer
Orderly Spanning Tree

Realizer
Orderly Spanning Tree

Realizer
Conclusion

necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same path $P_i$ ($1 \leq i \leq 3$).
**Theorem**

$G$ : plane graph with each degree $\geq 3$

(a) - (f) are equivalent with each other.

(a) $G$ has a **canonical decomposition**.
(b) $G$ has a **realizer**.
(c) $G$ has a **Schnyder labeling**.
(d) $G$ has an **outer triangular convex grid drawing**.
(e) $G$ has an **orderly spanning tree**.
(f) **necessary and sufficient condition**
   - $G$ is internally 3-connected.
   - $G$ has no separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$).
Corollary

$G$: plane graph with each degree $\geq 3$

If a plane graph $G$ satisfies the necessary and sufficient condition, then one can find the followings in linear time:

(a) canonical decomposition,
(b) realizer,
(c) Schnyder labeling,
(d) orderly spanning tree, and
(e) outer triangular convex grid drawing of $G$ having the size $(n-1) \times (n-1)$.

not 3-connected
The remaining problem is to characterize the class of plane graphs having convex grid drawings such that the size is \((n - 1) \times (n - 1)\) and the outer face is not always a triangle.
END
(ost1) For each edge not in the tree $T$, none of the endpoints is an ancestor of the other in $T$.

(ost2) For each leaf other than $u_\alpha$ and $u_\beta$, neither $N_2$ nor $N_4$ is empty.
- \(G\) is internally 3-connected
- \(G\) has no separation pair \(\{u, v\}\) s.t. both \(u\) and \(v\) are on the same \(P_i\) \((1 \leq i \leq 3)\)

\((cd2)\) Each \(G_k\) \((1 \leq k \leq h\) is internally 3-connected.

\((cd3)\) All the vertices in each \(V_k\) \((2 \leq k \leq h-1)\) are outer vertices of \(G_k\), and either (a) or (b).
• $G$ is internally 3-connected
• $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

$G$ has a canonical decomposition

(cd2) Each $G_k$ ($1 \leq k \leq h$) is internally 3-connected.

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Case 1
• $G$ is internally 3-connected
• $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

$G$ has a canonical decomposition

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Case 1

(a)

(b)
• \( G \) is internally 3-connected
• \( G \) has no separation pair \( \{u, v\} \) s.t. both \( u \) and \( v \) are on the same \( P_i \) \((1 \leq i \leq 3)\)

\( G \) has a canonical decomposition

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**Case 2**

\( F \)

\( G_i \)

\( u_1 \)

\( u_2 \)

\( P \)

\( G_{k-1} \)

(a)

\( V_k \)

(b)
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

$(cd2)$ Each $G_k$ ($1 \leq k \leq h$) is internally 3-connected.

$(cd3)$ All the vertices in each $V_k$ ($2 \leq k \leq h - 1$) are outer vertices of $G_k$, and either (a) or (b).
- $G$ is internally 3-connected
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$G$ has a canonical decomposition

(cd2) Each $G_k$ ($1 \leq k \leq h$) is internally 3-connected.

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Case 2
• \( G \) is internally 3-connected
• \( G \) has no separation pair \( \{u, v\} \) s.t. both \( u \) and \( v \) are on the same \( P_i \) (1 \( \leq \) \( i \) \( \leq \) 3)

\[ \text{(cd2) Each } G_k (1 \leq k \leq h) \text{ is internally 3-connected.} \]

\[ \text{(cd3) All the vertices in each } V_k (2 \leq k \leq h - 1) \text{ are outer vertices of } G_k, \text{ and either (a) or (b).} \]
• $G$ is internally 3-connected
• $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

$G$ has a canonical decomposition

(cd2) Each $G_k$ ($1 \leq k \leq h$) is internally 3-connected.

(cd3) All the vertices in each $V_k$ ($2 \leq k \leq h - 1$) are outer vertices of $G_k$, and either (a) or (b).

**Case 2**

- $V_i$ and $V_k$ are outer vertices of $G_i$ and $G_{k-1}$ respectively.
- $u_1$ and $u_2$ are on the same $P_i$ ($1 \leq i \leq 3$).

(a)

(b)
• $G$ is internally 3-connected
• $G$ has no separation pair $\{u,v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

$G$ has a canonical decomposition

(cd2) Each $G_k$ ($1 \leq k \leq h$) is internally 3-connected.

(cd3) All the vertices in each $V_k$ ($2 \leq k \leq h-1$) are outer vertices of $G_k$, and either (a) or (b).

Case 2
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

$G$ has a canonical decomposition

1. $P$ connects two outer vertices $u$ and $v$.
2. $u, v$ is a separation pair of $G$.
3. $P$ lies on an inner face.
4. $P$ does not pass through any outer edge and any outer vertex other than $u$ and $v$. 
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

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$G$ has a canonical decomposition
• $G$ is internally 3-connected
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$G$ has a canonical decomposition

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3. $P$ lies on an inner face.
4. $P$ does not pass through any outer edge and any outer vertex other than $u$ and $v$. 
• \( G \) is internally 3-connected
• \( G \) has no separation pair \( \{u, v\} \) such that both \( u \) and \( v \) are on the same \( P_i \) (1 ≤ \( i \) ≤ 3)

\( G \) has a canonical decomposition

1. \( P \) connects two outer vertices \( u \) and \( v \).
2. \( u, v \) is a separation pair of \( G \).
3. \( P \) lies on an inner face.
4. \( P \) does not pass through any outer edge and any outer vertex other than \( u \) and \( v \).

plan of proof:
find a minimal chord-path \( P \), then choose such a vertex set.
3-connected plane graph
3-connected plane graph
3-connected plane graph