ON THE RELATIONSHIP BETWEEN THE GENUS AND THE CARDINALITY OF THE MAXIMUM MATCHINGS OF A GRAPH

Takao NISHIZEKI
Department of Electrical Communications, Faculty of Engineering, Tohoku University, Sendai, Japan 980

Received 25 April 1978

Lower bounds on the cardinality of the maximum matchings of graphs are established in terms of a linear polynomial of \( p, p^{(1)}, p^{(2)} \) and \( \gamma \) whose coefficients are functions of \( \kappa \), where \( p \) is the number of the vertices of a graph, \( p^{(i)} \) the number of the vertices of degree \( i \) \((i = 1, 2)\), \( \gamma \) the genus and \( \kappa \) the connectivity.

1. Preliminary definitions and lemmas

In this paper we deal with simple finite undirected graphs (no loops and no multiple edges). \( G = (V, E) \) denotes a graph with vertex set \( V \) and edge set \( E \). A matching of a graph is a set of nonadjacent edges, and a maximum matching, denoted by \( M(G) \), of \( G \) is one of maximum cardinality. \( n(G) \) denotes the number of unsaturated vertices (i.e., vertices with which no edge of a matching is incident) in \( M(G) \). Therefore

\[
|M(G)| = \frac{1}{2}(p - n(G)) \tag{1}
\]

where \( p = |V| \). The genus \( \gamma(G) \) of a graph \( G \) is the smallest genus among all closed orientable 2-manifolds on which \( G \) can be embedded, i.e., the minimum number of handles which must be added to a sphere so that \( G \) can be embedded on the resulting surface. Of course, if \( G \) is planar, \( \gamma(G) = 0 \).

\[
d_G(v) \tag{2}
\]

\( d_G(v) \) denotes the degree of a vertex \( v \) in \( G \), and \( \delta(G) \) the minimum degree of vertices in \( G \), i.e., \( \delta(G) = \min_{v \in V} d_G(v) \). The connectivity \( \kappa(G) \) of a graph \( G \) is the minimum number of vertices whose removal results in a disconnected or trivial graph. \( \gamma(G) \) and \( \kappa(G) \) are often written \( \gamma \) and \( \kappa \) in short, respectively. For a subset \( S \) of \( V \), let \( G - S \) denote the graph obtained from \( G = (V, E) \) by the removal of all the vertices in \( S \). An odd component is one with an odd number of vertices. \( \lfloor x \rfloor \) means the greatest integer \( \leq x \), and \( \lceil x \rceil \) the least integer \( \geq x \). We define for sets \( A_1 \) and \( A_2 \), \( A_1 + A_2 = A_1 \cup A_2 \) if \( A_1 \cap A_2 = \emptyset \). A singleton set \{a\} is denoted by "a" for the sake of convenience.
Let a graph $G = (V, E)$ have $p$ vertices and $q$ edges. Let $I$ be the set of all positive integers, $I_o$ be the set of all positive odd integers, and $I_e$ be the set of all positive even integers. Define for $i \in I$

$$V^{(i)} = \{ v \mid v \in V \text{ and } d_G(v) = i \},$$

and

$$p^{(i)} = |V^{(i)}|.$$

**Lemma 1.** If $G$ is a connected bipartite graph, then

$$q \begin{cases} = 2p - p^{(1)} - 2 = 0 & \text{if } p = 1, \\ = 2p - p^{(1)} - 1 = 1 & \text{if } p = 2, \\ \leq \min \{ 2p - p^{(1)} + 4\gamma(G) - 2, 2p + 4\gamma(G) - 4 \} & \text{if } p \geq 3. \end{cases}$$

**Proof.** Both (3a) and (3b) are obvious. We shall show (3c). Suppose that $p \geq 3$. It has been shown that $q \leq 2p + 4\gamma(G) - 4$ (see [4]). Thus we shall show that $q \leq 2p - p^{(1)} + 4\gamma(G) - 2$. Let $G'$ be the graph with $p'$ vertices and $q'$ edges obtained from $G$ by the removal of all the vertices of degree 1, i.e., $G' = G - V^{(1)}$. Then

$$p' = p - p^{(1)},$$

and

$$q' = q - p^{(1)}.$$  

Since $p \geq 3$, $p' \geq 1$.

*Case 1:* $p' = 1$ or 2. Since $G$ is a tree,

$$q = p - 1 = \begin{cases} 2p - p^{(1)} - 2 & \text{if } p' = 1, \\ 2p - p^{(1)} - 3 & \text{if } p' = 2. \end{cases}$$

*Case 2:* $p' \geq 3$. Since $G'$ is a connected bipartite graph with at least three vertices [4],

$$q' \leq 2p' + 4\gamma(G') - 4.$$  

Since $\gamma(G') \leq \gamma(G)$, by (4), (5) and (6)

$$q \leq 2p - p^{(1)} + 4\gamma(G) - 4.$$  

**Lemma 2.** For any graph $G = (V, E)$, there exists a subset $S$ of $V$ such that

(a) $t - s = n(G)$;

and

(b) $V^{(1)} \subset S + W$,

where (i) $s = |S|$, (ii) $t$ is the number of odd connected components of $G - S$, (iii) $G_i = (V_i, E_i)$ is the $i$th connected component of $G - S$, (iv) $J_i = \{ j \mid |V_j| = i \}$ and
Proof. Berge has shown that there exists a subset $S$ of $V$ which satisfies Condition (a) (see [2, 3]). Suppose that $S$ is selected so that $|V^{(1)} \cap (V - (S + W))|$ is minimum among such subsets. We shall show that $V^{(1)} \cap (V - (S + W)) = \emptyset$. Assume that $V^{(1)} \cap (V - (S + W)) \neq \emptyset$. Then there exist (i) a vertex $v \in V$ such that $d_G(v) = 1$ and $v \in V_j$, $|V_j| \geq 2$, and (ii) a vertex $u \in V_j$ which is adjacent with $v$ in $G_j$ (and in $G$). Here $d_G(u) \geq 2$. If $|V_j|$ is odd, then $G_j - u$ has at least two odd connected components, one of which is the graph with the vertex set $\{v\}$. If $|V_j|$ is even, then $G_j - u$ has at least one odd connected component. Therefore if $S$ is replaced by $S' = S + u$, then

$$
t' = t + 1,
$$
$$
s' = s + 1,
$$
and

$$
W' = W + v,
$$
where $t'$, $s'$ and $W'$ are defined, with respect to $S'$, in the same fashion as $t$, $s$ and $W$. Hence

$$
t' - s' \geq n(G),
$$
where in fact equality holds, and

$$
|V^{(1)} \cap (V - (S' + W'))| < |V^{(1)} \cap (V - (S + W))|.
$$

This is a contradiction.

2. Main theorems

In a previous paper [6], we have established lower bounds on the cardinality of the maximum matchings of planar graphs with a constraint on the minimum degree. In this paper we show relationships between the genus and the cardinality of the maximum matchings of a graph (not always planar and with no constraints on the minimum degree) which partly generalize the previous results.

Theorem 1. If $G = (V, E)$ is connected, then

$$
|M(G)| \begin{cases} 
\geq \left\lfloor \frac{1}{2}(p - p^{(1)} - p^{(2)} - 4\gamma(G) + 2) \right\rfloor & \text{if } p + 2p^{(1)} + 2p^{(2)} + 8\gamma(G) \geq 10, \\
= \left\lfloor \frac{1}{2}p \right\rfloor & \text{otherwise}.
\end{cases}
$$

Proof. We can easily prove the above claim by using (1) and the following:

$$
n(G) \leq \max \{1, \left\lfloor (p + 2p^{(1)} + 2p^{(2)} + 8\gamma(G) - 4) \right\rfloor\}. \quad (7)
$$
Suppose that (7) is not true, that is,

$$2 \leq n(G);$$  \(\text{(8a)}\)

and

$$\frac{1}{2}(p + 2p^{(1)} + 2p^{(2)} + 8\gamma(G) - 4) < n(G).$$  \(\text{(8b)}\)

S, t, s, V, J, t_i, and W are defined as in Lemma 2. By (8a)

$$s \geq 1.$$  \(\text{(9)}\)

Now

$$t_1 \geq 1,\quad \text{because, if } t_1 = 0, \text{ then } p \geq 3t + s \text{ and } n(G) = t - s \leq \frac{1}{3}(p - 4s) \leq \frac{1}{3}(p - 4) \text{ by (9), contradicting (8b).}\) Let

$$E_S = \{(v, w) | (v, w) \in E \text{ and } v, w \in S\},$$

and

$$V^* = \bigcup_{t \in T} \left( \bigcup_{i \in J_t} V_i \right) = V - S - W.$$

Let G' = G - V^* - E_S with p' vertices and q' edges be the graph obtained from G - V^* by removing all the edges in E_S. Note that G' may contain isolated vertices. By definition, it is obvious that G' is a bipartite graph with edges joining the vertices in W and in S. By (10), G' is nontrivial. Hence

$$p' = t_1 + s.$$  \(\text{(11)}\)

By the definition of genus,

$$\gamma(G') \leq \gamma(G).$$  \(\text{(12)}\)

Let p^{(1,0)} be the number of vertices v such that v \in S, d_C(v) = 1 and d_G(v) = 0. Among the t_i vertices in W of G', at most (p^{(1)} - p^{(1,0)}) vertices are of degree 1, at most p^{(2)} are of degree 2, and the remainder are of degree \(\geq 3\). Therefore

$$q' \geq 3t_i - 2(p^{(1)} - p^{(1,0)}) - p^{(2)}.$$  \(\text{(13)}\)

Since \(t = \sum_{t \in T} t_i\) and \(|V_j| \geq 3\) if \(j \in J_i\) and \(i \geq 3\),

$$p' \leq p - 3(t - t_i).$$  \(\text{(14)}\)

By (8b) and (14),

$$t_1 - s = n(G) - (t - t_i) > \frac{1}{3}(p' + 2p^{(1)} + 2p^{(2)} + 8\gamma(G) - 4).$$  \(\text{(15)}\)

By (11) and (15),

$$t_1 > \frac{1}{3}(2p' + p^{(1)} + p^{(2)} + 4\gamma(G) - 2).$$  \(\text{(16)}\)

By (13) and (16),

$$q' > 2p' - p^{(1)} + 4\gamma(G) - 2 + 2p^{(1,0)}.$$  \(\text{(17)}\)

Let G' contain k connected components, let G'_i = (V'_i, E'_i) with p'_i vertices and
\( q'_i \) edges be the \( i \)th connected component of \( G' \) \((i = 1, 2, \ldots, k)\), and let \( k_2 \) be the number of connected components such that \( p'_i = 2 \). Then \( k \geq 1 \) by (10), and by [1]

\[
\gamma(G') = \sum_{i=1}^{k} \gamma(G'_i).
\]

(18)

Since \( G'_i \) is a connected bipartite graph, by Lemma 1

\[
q'_i = \begin{cases} 
2p'_i - p'^{(1)}_i - 1 & \text{if } p'_i = 2, \\
2p'_i - p'^{(1)}_i + 4\gamma(G'_i) - 2 & \text{otherwise},
\end{cases}
\]

(19)

where \( p'^{(1)}_i \) denotes the number of the vertices of degree 1 in \( G'_i \). Since \( q' = \sum_{i=1}^{k} q'_i \) and \( p' = \sum_{i=1}^{k} p'_i \), by (12), (18) and (19)

\[
q' \leq 2p' - p'^{(1)} + 4\gamma(G) - (2k - k_2).
\]

(20)

where \( p'^{(1)} \) is the number of the vertices of degree 1 in \( G' \). Since \( k \geq k_2 \),

\[
P_{k=k_2=1}^k \begin{cases} 
1 & \text{if } k = k_2 = 1, \\
> 2 & \text{otherwise}
\end{cases}
\]

(21)

Let \( v \in V \) be a vertex such that \( d_G(v) = 1 \), i.e., \( v \in V^{(1)} \). Then, by Lemma 2, \( v \in (S + W) \). If \( v \in W \), then \( d_G(v) = 1 \). If \( v \in S \), then \( d_G(v) = 1 \) or 0. If \( k = k_2 = 1 \), then there exists a vertex \( u \in S \) such that \( d_G(u) = 1 \) and \( d_G(u) \geq 2 \). Therefore

\[
p'^{(1)} + p'^{(1,0)} \geq \begin{cases} 
p'^{(1)} + 1 & \text{if } k = k_2 = 1, \\
p'^{(1)} & \text{otherwise}
\end{cases}
\]

(22)

By (20), (21) and (22),

\[
q' \leq 2p' - p'^{(1)} + 4\gamma(G) - 2 + p'^{(1,0)}.
\]

This contradicts (17). Thus we have shown (7).

**Theorem 2.** If \( \kappa(G) \geq 2 \), then

\[
|M(G)| \begin{cases} 
\leq \lceil \frac{1}{3}(p - p^{(2)} - 4\gamma(G) + 4) \rceil & \text{if } p + 2p^{(2)} + 8\gamma(G) \geq 14; \\
\lceil \frac{1}{3}p \rceil & \text{otherwise}.
\end{cases}
\]

(23)

**Proof.** We can easily prove the above claim by using (1) and the following:

\[
n(G) \leq \max \{1, \lceil \frac{1}{3}(p + 2p^{(2)} + 8\gamma(G) - 8) \rceil \}.
\]

(23)

(23) can be established by using (3c) and the arguments similar to those used in the derivation of (7). Note that \( s \geq 2 \) and \( p'_i \geq 3 \). Left to the reader.

**Theorem 3.** If \( G \) is a connected nonplanar graph, and \( \delta(G) \geq 2 \), then

\[
|M(G)| \geq \lceil \frac{1}{3}(p - p^{(2)} - 4\gamma(G) + 4) \rceil,
\]
\[ \gamma(G) \geq \left\lfloor \frac{1}{2}(P - p^{(2)} - 3|M(G)|) \right\rfloor + 1. \]

**Proof.** Similar to the proof of Theorem 2. Left to the reader.

**Theorem 4.** If \( \kappa(G) \geq 4 \), then
\[
|M(G)| \begin{cases} \left\lfloor \frac{1}{2}(P - 4\gamma/(\kappa - 2) + \kappa - 2) \right\rfloor & \text{if } 4\gamma \geq \kappa - 2; \\ \left\lceil \frac{1}{2}P \right\rceil & \text{otherwise}, \end{cases}
\]
that is,
\[
|M(G)| = \min \{ \left\lfloor \frac{1}{2}P \right\rfloor, \left\lfloor \frac{1}{2}(P - 4\gamma/(\kappa - 2) + \kappa - 2) \right\rfloor \}.
\]

**Proof.** We can prove (24) by using (1) and the following:
\[ n(G) \leq \max \{ 1, 4\gamma/(\kappa - 2) - \kappa + 2 \}. \] (26)

Suppose that (26) is not true, that is,
\[ 2 < n(G); \] (27a)
and
\[ 4\gamma/(\kappa - 2) - \kappa + 2 < n(G). \] (27b)

\( S, t, s, V, J, \) and \( t, s, V_{i}, J_{i} \) are defined as in Lemma 2. Let
\[ E_{S} = \{(u, w) \mid (u, w) \in E, \text{ and } u, w \in S \}, \]
and
\[ G_{b} = G - E_{S} - \bigcup_{i \in I_{1}} \bigcup_{j \in J_{i}} V_{i}, \]
where isolated vertices, if they appear, are removed. Define \( G' = (V', F') \) with \( p' \) vertices and \( q' \) edges to be the graph obtained from \( G_{b} \) by contracting all the vertices in \( V_{i} \) into one vertex, say \( x_{j} \), for every \( j \in \bigcup_{i \in I_{1}} J_{i} \). Then \( G' \) is a bipartite graph, and
\[ p' = t + s'. \] (28)
where \( s' = |S \cap V'| \leq s, \) and \( t = \sum_{i \in I_{1}} t_{i}. \) By (27a) and the definition of \( \kappa, \)
\[ d_{G'}(x_{j}) \geq \kappa(G), \] (29)
for every \( j \in \bigcup_{i \in I_{1}} J_{i}. \) By (29), each connected component of \( G' \) has at least five vertices. Therefore by Lemma 1,
\[ q' < 2p' + 4\gamma(G') - 4. \] (30)
On the other hand, by (29)
\[ q' \geq \kappa t. \] (31)
Since \( n(G) = t-s \), by (27b)

\[
(k-2)t > 2s + 4\gamma(G) + (\kappa - 4)s - (k-2)^2.
\]

(32)

Since \( s \geq k \geq 4 \),

\[
(k-2)t > 2s + 4\gamma(G) - 4.
\]

(33)

By (28), (31) and (33)

\[
q' > 2p' + 4\gamma(G) - 4.
\]

(34)

Here \( \gamma(G) \geq \gamma(G') \) because the genus of a subcontraction of a graph \( G \) does not exceed the genus of \( G \) (see [5]). Therefore (34) contradicts (30).

Note that Theorem 4 is a generalization of the property that \( |M(G)| = \lfloor \frac{2}{3}p \rfloor \) if \( G \) is a 4-connected planar graph, which is an immediate consequence of Tutte’s result that every 4-connected planar graph has a hamiltonian cycle [9]. From Theorems 1, 2 and 4 we obtain the following corollary which has been shown in [6].

**Corollary.** If \( G \) is planar and \( \delta(G) \geq 3 \), then

\[
|M(G)| \begin{cases} 
\geq \lceil \frac{2}{3}(p+2) \rceil & \text{if } \kappa(G) = 1 \text{ and } p \geq 10, \\
\geq \lceil \frac{2}{3}(p+4) \rceil & \text{if } \kappa(G) = 2 \text{ or } 3, \text{ and } p \geq 14, \\
\geq \lfloor \frac{1}{3}p \rfloor & \text{otherwise.}
\end{cases}
\]

3. **Examples**

Let \( C_i \) denote a cycle with \( i \) vertices, \( K_i \) the complete graph with \( i \) vertices, and \( K_{ij} \) the complete bipartite graph with \( i \) black and \( j \) white vertices and with \( ij \)

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
</tr>
<tr>
<td>----------</td>
</tr>
<tr>
<td>( C_p )</td>
</tr>
<tr>
<td>( K_{1p} )</td>
</tr>
<tr>
<td>( K_{2p} )</td>
</tr>
</tbody>
</table>
| \( K_{3p} \) | 3 | 3 | 0 | 0 | \( \lceil \frac{2}{3}(p-5) \rceil \) (see [7]) | \( \lfloor \frac{2}{3}(p-p^{(2)} - 4\gamma + 4) \rfloor = \begin{cases} 2 & \text{if } p \equiv 2 \pmod{4} \\
3 & \text{otherwise (Theorem 3)} \end{cases} \) |
| \( K_{\kappa p} \) | \( \kappa \) | \( \kappa \) | 0 | 0 | \( \lceil 2\kappa(p-2) - (p - \kappa - 2) \rceil \) (see [7]) | \( \lfloor 2\kappa(p-2) - (p - \kappa - 2) \rfloor \) (Theorem 4) |
| \( K_p \) | \( p-1 \) | \( p-1 \) | 0 | 0 | \( \lceil \frac{2}{3}(p-3)(p-4) \rceil \) (see [1]) | \( 4\gamma < \kappa(k-2) \) (Theorem 4) |
edges joining a black and a white vertex. The properties of such graphs are exhibited in Table 1. By the examples in Table 1 and in [6] it can be shown that Theorems 1, 2, 3, and 4 are best possible in the sense that neither the coefficients of \( p, p', p'' \), and \( \gamma(G) \), nor the additive constant terms can be improved. It is interesting that the equality in Theorem 4 holds for every complete and complete bipartite graph of connectivity \( \geq 4 \) as shown in Table 1.

Acknowledgement

This research was done during the author's stay at Carnegie-Mellon University with support from Murata Oversea Scholarship Foundation in Kyoto, Japan, and partly from Grant DAAG 29-77-G-0024. He would like to express his gratitude to Professor R. J. Duffin, Professor N. Saito, Professor M. I. Shamos and Mr. I. Baybars for their suggestions and encouragements.

References

[4] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1972 (revised)).