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AN APPROXIMATION ALGORITHM FOR THE HAMILTONIAN WALK PROBLEM ON MAXIMAL PLANAR GRAPHS

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A hamiltonian walk of a graph is a shortest closed walk that passes through every vertex at least once, and the length is the total number of traversed edges. The hamiltonian walk problem in which one would like to find a hamiltonian walk of a given graph is NP-complete. The problem is a generalized hamiltonian cycle problem and is a special case of the traveling salesman problem. Employing the techniques of divide-and-conquer and augmentation, we present an approximation algorithm for the problem on maximal planar graphs. The algorithm finds, in $O(p^2)$ time, a closed spanning walk of a given arbitrary maximal planar graph, and the length of the obtained walk is at most $\frac{3}{2}(p-3)$ if the graph has p (≥ 9) vertices. Hence the worst-case bound is $\frac{3}{2}$.

1. Introduction

A hamiltonian walk of a graph is a shortest closed walk that passes through every vertex at least once, and the length of a hamiltonian walk is the total number of edges traversed by the walk [7]. The hamiltonian walk problem in which one would like to find a hamiltonian walk of a given graph would arise in situations where it is necessary to periodically traverse a network or data structure in a way as to visit all vertices and minimize the length of the traversal.

The hamiltonian walk problem is a generalization of the hamiltonian cycle problem in which one would like to determine if a given graph contains a hamiltonian cycle. It is well known that the hamiltonian cycle problem is NP-complete. Furthermore Garey, Johnson and Tarjan [6] have shown that the hamiltonian cycle problem is NP-complete even when restricted to 3-connected, cubic, planar graphs. Hence the hamiltonian walk problem is also NP-complete even when restricted to the same class. On the other hand, Christfides [4] has developed a polynomial-time approximation algorithm with a worst-case bound of $\frac{3}{2}$ for the traveling salesman problem in which the distance between vertices satisfies the triangle inequality. The

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hamiltonian walk problem is a special case of the traveling salesman problem, in which the distances are the shortest path lengths in a graph. Therefore we can use the algorithm of Christofides to find in $O(p^3)$ time a closed spanning walk of a graph whose length is smaller than $\frac{3}{2}$ times the length of a shortest one, where p denotes the number of vertices in a graph.

In this paper we present a new approximation algorithm for the hamiltonian walk problem on maximal planar graphs. Our algorithm is more efficient than the method employing the Christofides' algorithm, and is completely different from it. Given a maximal planar graph with p (≥ 9) vertices, our algorithm finds, in $O(p^2)$ time, a closed spanning walk of the graph whose length is at most $\frac{3}{2}(p-3)$. Hence it has a worst-case bound of $\frac{3}{2}$ which is same as his. However it should be noted that the Christofides' method does not always produce a closed spanning walk of length at most $\frac{3}{2}(p-3)$ if the given graph is not hamiltonian(, since in this case the shortest closed walk has length greater than p). We will employ, in our algorithm, two techniques: divide-and-conquer and augmentation. The algorithm is based on two known results: one is our previous result that a maximal planar graph with p vertices always contains either a hamiltonian cycle or a closed spanning walk of length at most $\frac{3}{2}(p-3)$ [2]; the other is Whitney's result that every 4-connected, maximal planar graph has a hamiltonian cycle [12]. We conjecture that the hamiltonian walk problem remains NP-complete even if we restrict ourselves to the class of maximal planar graphs.

2. Terminology and basic results

We proceed to some basic definitions. An *(undirected simple) graph* $G=(V,E)$ consists of a set V of vertices and a set E of edges. Throughout this paper p denotes the number of vertices of G, i.e., $p=|V|$. A *walk of length* k of G is a sequence $v_0e_1v_1e_2\cdots e_kv_k$, whose term are alternately, vertices and edges, such that the end-vertices of edge e_i are v_{i-1} and v_i, for each $1\leq i\leq k$. The length of a walk W is denoted by $l(W)$. The walk W is a *closed spanning walk* of G if $v_0=v_k$ and every vertex of G appears in the sequence at least once. A *hamiltonian walk* of G is a closed spanning walk of minimum length. For a connected graph G, let $h(G)$ denote the length of a hamiltonian walk of G. Clearly $p\leq h(G)\leq 2(p-1)$. A *cycle* is a closed walk whose vertices are all distinct. A *hamiltonian cycle* of G is a closed spanning walk of length p, i.e., a cycle that passes through every vertex of G exactly once. A graph is *hamiltonian* if it contains a hamiltonian cycle. A *maximal planar graph* is a planar graph to which no edge can be added without losing planarity. Note that every maximal planar graph G is connected and every face of G is a triangle. A triangle of a maximal planar graph is called a *nonface triangle* if it is not a boundary of a face. A maximal planar graph with p (≥ 5) vertices has no nonface triangles if and only if it is 4-connected. For a graph $G=(V,E)$ and a subset V' of V, $G-V'$ denotes the graph obtained from G by deleting all vertices in V'. A singleton set $\{v\}$ is simply denoted by 'v'. A multiset is a set with a function mapping the elements of the set into the positive integers, to indicate that an element may appear more than once. We sometimes represent a walk by the multiset of edges traversed by it. A walk W', i.e., a sequence of edges and vertices, can be easily constructed from the multiset of edges of a walk W, (possibly $W' \neq W$). Note that this can be done by any algorithm for finding an eulerian walk of an eulerian graph. We refer to [1] or [9] for all undefined terms.

We next present several lemmas. Generalizing Whitney's result [12], Tutte has shown that every 4-connected planar graph has a hamiltonian cycle [11]. Employing the proof technique used by Tutte, Gouyou-Beauchamps has given an $O(p^3)$ algorithm for finding a hamiltonian cycle in a 4-connected planar graph G [8]. If G is maximal planar, we can improve the time-complexity as follows.

Lemma 1. *There is a* $O(p^2)$ *time-algorithm for finding a hamiltonian cycle in a* 4-*connected maximal planar graph* G *with* p *vertices.*

Remark. It is not difficult to implement a recursive algorithm for finding a hamiltonian cycle of a 4-connected maximal planar graph G in $O(p^2)$ time, completely based on the inductive proof of Whitney [12] ensuring its existence. On the other hand, Asano, Kikuchi and Saito have recently obtained an $O(p)$ algorithm for the same purpose. (T. Asano, S. Kikuchi and N. Saito, An efficient algorithm to find a Hamiltonian circuit in a 4-connected maximal planar graph, in: N. Saito and T. Nishizeki, eds., Graph Theory and Algorithms, Lecture Notes in Computer Science 108 (Springer-Verlag, Berlin, 1981) 182–195.)

Lemma 2. (a) *Every maximal planar graph with ten or fewer vertices contains a hamiltonian cycle* [3,12].

(b) *Every nonhamiltonian, maximal planar graph with* 11 *vertices has a hamiltonian walk of length* 12. (*It is implicit in* [3] *that every such graph is isomorphic to a certain graph depicted in Fig.* 1 *of* [2].)

Lemma 3. [2] *Let* xyz *be any* (*triangular*) *face of a maximal planar graph* $G=(V,E)$ *with* p *vertices, where* $x,y,z \in V$.

(a) *If* $p=5$ *or* 6, *then at least one of the three graphs* $G-\{x,y\}$, $G-\{y,z\}$ *and* $G-\{z,x\}$ *contains a hamiltonian cycle.*

(b) *If* $p=7$ *or* 8, *then* (i) *at least one of* $G-\{x,y\}$, $G-\{y,z\}$ *and* $G-\{z,x\}$ *contains a hamiltonian cycle, or* (ii) $G-x$, $G-y$ *and* $G-z$ *all have hamiltonian cycles.*

Using the dynamic programming, one can obtain an $O(p \cdot 2^p)$ algorithm for determining if a given graph contains a hamiltonian cycle. The algorithm requires a constant time if p is a constant. Let HAMILTON(G) be such an algorithm which determines if a given planar graph with 11 *or fewer* vertices contains a hamiltonian cycle, and moreover finds a hamiltonian walk in constant time. This algorithm will be used in the next section.

Lemma 4. *Given a connected graph $G=(V,E)$ with p vertices and given a cycle C of length c in G, one can find a closed spanning walk W of G such that $l(W)\le 2p-c$, in $O(|E|)$ time.*

Proof. Contract all the vertices on C into one vertex, and find a (spanning) tree of the obtained graph. If C is the set of edges of the cycle C and T the set of edges of the tree, then the multiset $W=C+T+T$ is a closed spanning walk of G which traverses twice each edge of the tree and once each edge of C. Clearly the length of W is $2p-c$. □

For a nonface triangle T of a maximal plane graph G, let $G_{TI}=(V_{TI},E_{TI})$ denote the induced subgraph of G inside T, and $G_{TO}=(V_{TO},E_{TO})$ the induced subgraph of G outside T. Specifically if $T=xyz$ ($x,y,z\in V$), $U'(T)$ is the set of vertices lying inside T, and $U''(T)$ is the set of vertices outside T, then G_{TI} is the subgraph of G induced by the vertex set $\{x,y,z\}\cup U'(T)$, i.e., $G_{TI}=G-U''(T)$, and G_{TO} is the subgraph of G induced by the vertex set $\{x,y,z\}\cup U''(T)$, i.e., $G_{TO}=G-U'(T)$. Let $p_{TI}=|V_{TI}|$ and $p_{TO}=|V_{TO}|$. The following lemma plays a crucial role in the design of our algorithm. The precise description of Algorithm LCYCLE and the proof of Lemma 5 will appear in Section 4.

Lemma 5. *For a maximal planar graph G with p (≥ 11) vertices such that either $p_{TI}=4$ or $p_{TO}=4$ for each nonface triangle T of G, Algorithm* LCYCLE *finds a cycle C of length $l(C)\ge\frac{1}{2}(p+9)$, in $O(p^2)$ time.*

3. Approximation algorithm HWALK

In this section we present a polynomial-time algorithm for finding a closed spanning walk W with $l(W)\le\max\{p,\frac{3}{2}(p-3)\}$ of a given maximal planar graph G with p vertices. In the algorithm we will employ the divide-and-conquer technique: if a given maximal planar graph G has a nonface triangle T satisfying a certain condition, then (i) divide G into two smaller maximal planar graphs G_{TI} and G_{TO}, (ii) recursively call the algorithm with respect to G_{TI} and G_{TO}, and (iii) combine the closed spanning walks of G_{TI} and G_{TO} into a closed spanning walk of the whole graph G.

The Algol-like procedure HWALK depicted in Fig. 1 takes as input a maximal planar graph and returns a closed spanning walk of the graph represented by a multiset of edges.

We can show that HWALK is a polynomial-time algorithm with a worst-case bound of $\frac{3}{2}$, establishing the following theorem. Remember that $h(G)\ge p$ for every connected graph G.

Theorem 1. *For a maximal planar graph G with p vertices, Algorithm* HWALK

```
    procedure HWALK(G):
    begin
1     if p ≤ 11 then return a hamiltonian walk W of G which
        can be found by the Algorithm HAMILTON;
2     if G has no nonface triangle (i,e., G is 4-connected)
        then return a hamiltonian cycle W of G which can be
          found by the algorithm in Lemma 1
      else
3       if either p_TI = 4 or p_TO = 4 for every nonface triangle
          T of G then
            begin
4             find, in G, a cycle C of length ℓ(C) ≥ (p+9)/2
               by Algorithm LCYCLE (Lemma 5);
5             return a closed spanning walk constructed
               from C by the algorithm in Lemma 4
            end
        else
            begin
6             let T = xyz (x,y,z∈V) be a nonface triangle
               such that p_TI,p_TO ≥ 5;
              comment p = p_TI + p_TO - 3 ≥ 12;
7             wlg  assume that p_TI ≤ p_TO otherwise interchange
               roles of G_TI and G_TO in
                begin
8                 if p_TO ≥ p_TI ≥ 9 then return
                   HWALK(G_TI) + HWALK(G_TO);
9                 if p_TI = 7 or 8 and p_TO ≥ 9 then
                    begin
                      comment By Lemma 3(b) at least one of
                       G_TI-x, G_TI-y, G_TI-z, G_TI-{x,y}, G_TI-{y,z}
                        and G_TI-{z,x} is hamiltonian;
10                    find a hamiltonian cycle C_I of one
                       of the six graphs in the above
                        comment;
11                    return C_I + HWALK(G_TO)
                    end;
12                if p_TI = 7 or 8 and p_TO = 7 or 8 then
13                  if either G_TI-{x,y}, G_TI-{y,z} or
                     G_TI-{z,x} contains a hamiltonian
                      cycle C_I then return C_I + HWALK(G_TO)
                    else
14                    if either G_TO-{x,y}, G_TO-{y,z} or
                       G_TO-{z,x} contains a hamiltonian
                        cycle C_O then return HWALK(G_TI) + C_O
```

Fig. 1. Algorithm HWALK.

```
15                  else return
                      HAMILTON(G_TI-x) + HAMILTON(G_TO-y);
16              if p_TI = 5 or 6 then
                  begin
                    comment Either G_TI-{x,y}, G_TI-{y,z}
                     or G_TI-{z,x} is hamiltonian;
17                  wlg G_TI-{x,y} is hamiltonian in
                       find a hamiltonian cycle C_I of
                        G_TI-{x,y};
18                  return C_I + HWALK(G_TO)
                  end
              end
          end
      end
```

Fig. 1. Algorithm HWALK (cont.).

finds, in $O(p^2)$ *time, a closed spanning walk* W *of* G *such that*

$$l(W)\begin{cases}\le \frac{3}{2}(p-3) & \text{if } p\ge 11; \quad (1a)\\ =p & \text{otherwise.} \quad (1b)\end{cases}$$

Proof. We first prove correctness by induction on the number p of vertices of G. If $p\le 11$, then the algorithm finds a hamiltonian walk W in line 1, and Lemma 2 implies that $l(W)$ satisfies (1). For the inductive step, we assume that the Algorithm correctly finds a closed spanning walk W satisfying (1) in any maximal planar graph with less than p (≥ 12) vertices. Let G be a maximal planar graph with p vertices. If G has no nonface triangle (i.e., G is a 4-connected), then the Algorithm returns in line 2 a hamiltonian cycle W (by the algorithm in Lemma 1) which clearly satisfies (1). If either $p_{TI}=4$ or $p_{TO}=4$ for each nonface triangle T of G, then the Algorithm LCYCLE called in line 4 finds a cycle C of G such that $l(C)\ge \frac{1}{2}(p+9)$ (Lemma 5), and the Algorithm HWALK returns in line 5 a closed spanning walk W which is constructed from C of G. By Lemma 4 we have that

$$l(W)\le 2p-l(C)\le \tfrac{3}{2}(p-3).$$

In the remaining case in which there exists a nonface triangle T such that $p_{TI}, p_{TO}\ge 5$, we can assume without loss of generality that $p_{TI}\le p_{TO}$: otherwise interchange roles of G_{TI} and G_{TO}. Note that both G_{TI} and G_{TO} are maximal planar graphs with less than p vertices. If $p_{TO}\ge p_{TI}\ge 9$, then recursively calling itself the Algorithm finds closed spanning walks $W_I=\mathrm{HWALK}(G_{TI})$ of G_{TI} and $W_O=\mathrm{HWALK}(G_{TO})$ of G_{TO} in line 8. Clearly the multiset $W=W_I+W_O$ returned in line 8 represents a closed spanning walk of G. We shall show $l(W)\le \frac{3}{2}(p-3)$. It can be shown that

$$l(W_I)\le \tfrac{3}{2}(p_{TI}-3) \quad\text{and}\quad l(W_O)\le \tfrac{3}{2}(p_{TO}-3).$$

If $p_{TI}\ge 11$, then by the inductive hypothesis $l(W_I)\le \frac{3}{2}(p_{TI}-3)$; otherwise, i.e., if

$p_{TI}=9$ or 10, then HWALK finds a hamiltonian cycle W_I, so $l(W_I)=p_{TI}\le\frac{3}{2}(p_{TI}-3)$; the proof for the case of G_{TO} is similar. Since $p=p_{TI}+p_{TO}-3$, we have

$$l(W)=l(W_I)+l(W_O)\le\tfrac{3}{2}(p_{TI}-3)+\tfrac{3}{2}(p_{TO}-3)=\tfrac{3}{2}(p-3).$$

If $p_{TI}=7$ or 8 and $p_{TO}\ge 9$ (in line 9), then by Lemma 3(b) at least one of $G_{TI}-x$, $G_{TI}-y$, $G_{TI}-z$, $G_{TI}-\{x,y\}$, $G_{TI}-\{y,z\}$ and $G_{TI}-\{z,x\}$ has a hamiltonian cycle, say C_I. Clearly $l(C_I)=p_{TI}-1$ or $p_{TI}-2$. Let $W_O=\text{HWALK}(G_{TO})$, i.e., a closed spanning walk of G_{TO} obtained by recursively calling HWALK for G_{TO}. Then since $9\le p_{TO}<p$, $l(W_O)\le\frac{3}{2}(p_{TO}-3)$ as shown above. Hence $W=C_I+W_O$ is a closed spanning walk of G and

$$l(W)\le p_{TI}-1+\tfrac{3}{2}(p_{TO}-3)\le\tfrac{3}{2}(p-3).$$

Using Lemma 3 and the inductive hypothesis we can easily establish the correctness for the remaining cases.

We next prove that the total amount of time spent by HWALK is at most $O(p^2)$. Algorithm HAMILTON used in line 1 etc. determines whether a given graph with 11 or fewer vertices is hamiltonian or not and returns a hamiltonian walk (or cycle), both in constant time (, since $p\le 11$). By Lemma 1 the algorithm used in line 2 requires $O(p^2)$ time. By Lemma 5 LCYCLE called in line 4 requires $O(p^2)$ time, and by Lemma 4 a closed spanning walk of G can be constructed, in $O(p)$ time, from a cycle found by LCYCLE. Note that $O(|E|)=O(p)$ since G is planar. It shall be noted that if a maximal planar graph G contains a vertex w such that both end-vertices u and v of an edge $e=(u,v)$ are adjacent to w and the triangle uvw is not a face, then uvw is a nonface triangle of G. Using this fact, one can determine, in $O(p)$ time, whether G contains a nonface triangle with e as a boundary edge. Since $O(|E|)=O(p)$, one can find all nonface triangles of G in $O(p^2)$ time. It can be easily shown by induction on p that every maximal planar graph with p vertices contains at most $p-4$ nonface triangles. Hence one can determine all p_{TI} and p_{TO} for all nonface triangles T of G in $O(p^2)$ time. Moreover one can determine the inclusion relation among all nonface triangles of G. The relation is represented by a rooted tree R such that

(i) the root of R corresponds to the exterior face triangle of G;

(ii) each vertex of R except the root corresponds to a nonface triangle of G; and

(iii) a directed edge joins vertex x to vertex y in R if and only if the nonface triangle of G corresponding to y is an outmost triangle contained in the triangle corresponding to x.

If T is a nonface triangle of G, every nonface triangle except T is also a nonface triangle of G_{TI} or G_{TO}. Once one finds all nonface triangles T of G together with p_{TI} and p_{TO} and determines the inclusion relation among them, one can update such information for G_{TI} and G_{TO} in $O(p)$ time. Hence it is not difficult to implement HWALK so that the time $T(p)$ spent for a graph with p vertices satisfies

$$T(p)\le\max\{k_1p^2,\ T(p_{TI})+T(p_{TO})+k_2p,\ k_3+T(p_{TO})\},$$

where k_1, k_2 and k_3 are constants. Noting that $p = p_{TI} + p_{TO} - 3$, and solving the above equation, we have that $T(p) \leq O(p^2)$, establishing Theorem 1. □

4. Algorithm LCYCLE

In this section we describe a polynomial-time algorithm LCYCLE and prove Lemma 5. Given a maximal planar graph G with p (≥ 11) vertices such that either $p_{TI} = 4$ or $p_{TO} = 4$ for each nonface triangle T of G, Algorithm LCYCLE returns a cycle C with $l(C) \geq \frac{1}{2}(p+9)$ in $O(p^2)$ time.

In order to find a required cycle in a given graph we will employ a kind of augmentation: whenever the graph contains a vertex x off a currently obtained cycle C satisfying a certain requirement, some edges of C are replaced by appropriate edges off C so that x becomes a vertex on the newly obtained cycle and the length of the cycle increases by one. Consider the configurations depicted in Fig. 2, where C is written as $C = v_0 v_1 v_2 \cdots v_0$.

(I) Fig. 2(a) shows a configuration in which G has a vertex x off C which is adjacent to the endvertices v_1 and v_2 of an edge (v_1, v_2) on C. It shall be noted that probably $v_j = v_3$ where v_j is the third vertex to which x is adjacent. (It will be known that every vertex off C is of degree 3.) Clearly cycle $C' = v_0 v_1 x v_2 v_3 \cdots v_0$ of G is longer than C.

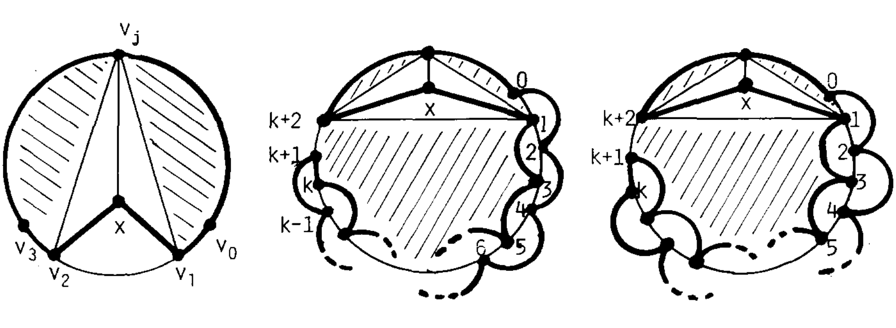
(II) Figs. 2(b) and (c) show configurations in which for some integer $k \geq 1$,

(i) $(v_{i-1}, v_{i+1}) \in E$ for each i, $1 \leq i \leq k$, and

(ii) a vertex x off C is adjacent to v_1 and v_{k+2}.

For simplicity vertex v_i is indicated by 'i' in Figs. 2(b) and (c). If k is odd, then clearly cycle

$$C' = v_0 v_2 v_4 \cdots v_{k-1} v_{k+1} v_k v_{k-2} \cdots v_3 v_1 x v_{k+2} \cdots v_0$$



(a) Type I (b) Type II with odd k (c) Type II with even k

Fig. 2. Configurations of type I and II. (An old cycle is drawn by lines on a circle, and a new cycle is drawn by thick lines.)

of G is longer than C. (See Fig. 2(b).) If k is even, then cycle

$$C' = v_0 v_2 v_4 \cdots v_k v_{k+1} v_{k-1} \cdots v_3 v_1 x v_{k+2} \cdots v_0$$

of G is longer than C. (See Fig. 2(c).)

The configuration illustrated in Fig. 2(a) is called *of type* I, and both in Figs. 2(b) and (c) together with symmetric ones are called *of type* II. Note that a configuration of type I can be regarded as a special case of type II with $k=0$.

For an illustration we depict in Fig. 3 a maximal planar graph $G=(V,E)$ with $V=\{0,1,2,\ldots,16\}$. Cycle $C=012\cdots(14)0$ is drawn by lines on a circle. Vertices 15 and 16 off C are of degree 3. G contains a configuration of type I with respect to vertex 16: $(16,0),(16,1)\in E$. G also contains a configuration of type II with respect to vertex 15: $(6,8),(7,9),(8,10),(15,7),(15,11)\in E$ $(k=3)$. The new cycle

$$C' = 0(16)1234568(10)97(15)(11)(12)(13)(14)0$$

longer than C is drawn by thick lines.

Algorithm LCYCLE is depicted in Fig. 4. We assume that cycle C is written generically as $v_0 v_1 v_2 \cdots v_{l(C)-1} v_0$ at any stage of the algorithm.

We next present the proof of Lemma 5.

Proof of Lemma 5. In order to prove the correctness of Algorithm LCYCLE, it is sufficient to show that (i) if $p \le 16$, then G contains a cycle C of length $l(C) \ge \frac{1}{2}(p+9)$, and that (ii) if $p \ge 17$ and C is an arbitrary cycle of length $l(C) < \frac{1}{2}(p+9)$

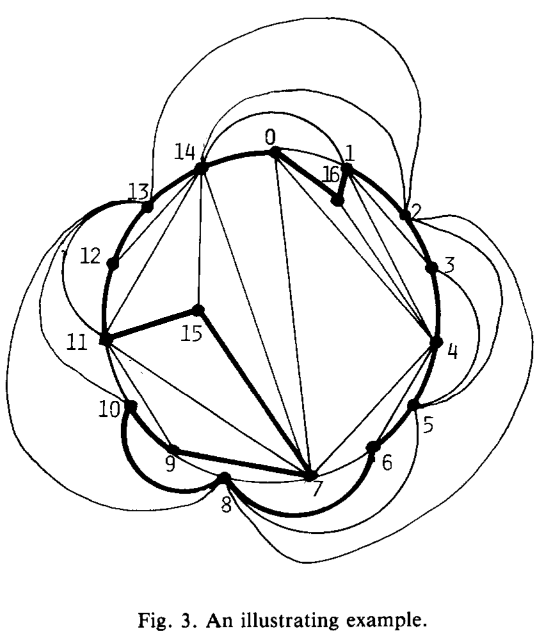


Fig. 3. An illustrating example.

```
   procedure LCYCLE(G):
1  if p ≤ 16 then return a cycle C of length ℓ(C) ≥ (p+9)/2
    found by any reasonable algorithm
   else
      begin
2        let V3 be the set of all vertices of degree 3;
3        G' ← G - V3;
         comment G' is a 4-connected maximal planar graph;
4        C ← a hamiltonian cycle of G' which can be found
               by the algorithm in Lemma 1;
5        ℓ(C) ← p - |V3|;
6        while ℓ(C) < (p+9)/2 do
            begin
7              if G contains a configuration of type I then
8                 wlg assume that vertex x off C is adjacent
                   to v1 and v2, the endvertices of edge
                    (v1,v2) on C otherwise rename the vertices
                    on C in
                    begin
9                      ℓ(C) ← ℓ(C) + 1;
10                     C ← v0v1xv2v3...v0
                    end
               else
                begin
                  comment G contains a configuration of type II;
11                wlg assume that for some k ≥ 1 (i) (vi-1,vi+1)∈E
                   for each i, 1 ≤ i ≤ k, and (ii) vertex x off C
                    is adjacent to v1 and vk+2 otherwise rename
                    vertices on C in
                    begin
12                     ℓ(C) ← ℓ(C) + 1;
13                     if k is odd then
                        C ← v0v2v4...vk-1vk+1vk...v3v1xvk+2...v0
                       else
14                      C ← v0v2v4...vkvk+1vk-1...v3v1xvk+2...v0
                    end
                end
            end
      end
```

Fig. 4. Algorithm LCYCLE.

such that every vertex off C has degree 3, then G contains a configuration of type I or II. In Section 4 of our previous paper [2] we showed via a lengthy argument that every maximal planar graph satisfying the requirement of Lemma 5 contains a cycle C of length $l(C) \geq \frac{1}{2}(p+9)$. Thus (i) above has been verified. Furthermore one can see that (ii) above is implicit in the arguments in Stages 1 and 2 of Section 4 of [2]. It shall be noted that the graph $G' = G - V_3$ in line 3 is 4-connected since G' contains no nonface triangles: otherwise G would contain a nonface triangle T with $p_{TI}, p_{TO} \geq 5$, contradicting the assumption of G. Thus every vertex off C is of degree 3 for the cycle C obtained in line 4. Once a vertex of G is inserted into C, it will be never deleted from C in the algorithm. Consequently, every vertex off C is of degree 3 at any stage of LCYCLE.

We next establish our claim on the time complexity of LCYCLE. If $p \leq 16$, one can find a cycle C of length $l(C) \geq \frac{1}{2}(p+9)$ in constant time. Therefore line 1 of LCYCLE requires O(1) time. Clearly lines 2 and 3 can be executed in $O(p)$ time. In line 4 we use the algorithm of Lemma 1 which is the most time consuming part of LCYCLE and requires $O(p^2)$ time. Clearly line 5 requires O(1) time. If edge $e = (v_1, v_2)$ is on C, (v_1, x) is an edge which is incident to v_1 and is clockwise or counterclockwise next to e in the plane embedding of G, and x is off C, then v_2 is adjacent to x, i.e., G contains a configuration of type I. Using the doubly linked adjacency lists for representing the plane embedding of G so that an edge embedded next to a given edge can be directly accessed, one can determine in O(1) time whether both endvertices of a given edge on C are adjacent to a vertex off C. Checking this for each edge on C, one can determine in $O(p)$ time whether G contains a configuration of type I. Note that one can embed a planar graph G on the plane in $O(p)$ time [10]. Similarly one can determine in $O(p)$ time whether G contains a configuration of type II. Each time a configuration of type I or II is found, cycle C is replaced by a longer one in lines 10, 13 and 14. Each replacement requires $O(p)$ time. Clearly each execution of line 9 or 12 requires O(1) time. Thus every execution of the loop of lines 7–14 requires $O(p)$ time. Since $l(C)$ increases by one after every execution of the loop, the loop is executed at most p times. Hence the total amount $L(p)$ of time spent by LCYCLE for a graph G satisfies $L(p) \leq O(p^2)$, so we have the lemma. □

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Note added in proof

Recently the conjecture mentioned at the end of Section 1 has been proved in: V. Chvátal, Hamiltonian cycles, Tech. Rep. 81-38, School of Compt. Sci., McGill Univ., Montreal, Canada (1981).

References

[1] A.V. Aho, J.E. Hopcroft and J.D. Ullman, The Design and Analysis of Computer Algorithms (Addison-Wesley, Reading, MA, 1974).

[2] T. Asano, T. Nishizeki and T. Watanabe, An upper bound on the length of a hamiltonian walk of a maximal planar graph, J. of Graph Theory 4 (1980) 315-336.
[3] D. Barnette and E.J. Jucovič, Hamiltonian circuits on 3-polytopes, J. Combin. Theory 2 (1970) 54-59.
[4] N. Christofides, Worst-case analysis of a new heuristic for the traveling salesman problem, Managing Sciences Research Report, 388, Carnegie-Mellon University (1976).
[5] M.R. Garey and D.S. Johnson, Approximation algorithm for combinatorial problems: An annoted bibiography, in: J.F. Traub, ed., Algorithms and Complexity: Recent Results and New Directions (Academic Press, New York, 1976) 41-52.
[6] M.R. Garey, D.S. Johnson and R.E. Tarjan, The planar Hamiltonian circuit problem is NP-complete, SIAM J. Comput. 5 (1976) 704-714.
[7] S.E. Goodman and S.T. Hedetniemi, On Hamiltonian walk in graphs, SIAM J. Comput. 3 (1974) 214-221.
[8] D. Gouyou-Beauchamps, Un algorithme de recherche de circuit Hamiltonien dans les graphes 4-connexes planaires, in: J.C. Bermond, J.C. Fournier, M. Las Vergnas and D. Scotteau, eds., Colloques Internationaux CNRS, No. 260 — Problems Combinatoires et Théorie des Graphes (1978) 185-187.
[9] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).
[10] J. Hopcroft and R. Tarjan, Efficient planarity testing, J. Assoc. Comput. Mach. 21 (1974) 549-568.
[11] W.T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956) 99-116.
[12] H. Whitney, A theorem on graphs, Annals Math. 32 (1931) 378-390.