Decompositions to Degree-Constrained Subgraphs
Are Simply Reducible to Edge-Colorings

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The degree-constrained subgraphs decomposition problem, such as an \( f \)-coloring, an \( f \)-factorization, and a \([g,f]\)-factorization, is to decompose a given graph \( G=(V,E) \) to edge-disjoint subgraphs degree-constrained by integer-valued functions \( f \) and \( g \) on \( V \). In this paper we show that the problem can be simply reduced to the edge-coloring problem in polynomial-time. That is, for any positive integer \( k \), we give a polynomial-time transformation of \( G \) to a new graph such that \( G \) can be decomposed to at most \( k \) degree-constrained subgraphs if and only if the new graph can be edge-colored with \( k \) colors.

1. INTRODUCTION

A degree-constrained subgraphs decomposition is to decompose a given graph \( G=(V,E) \) to edge-disjoint spanning subgraphs satisfying a degree constraint. For example, an \( f \)-coloring decomposes \( G \) to subgraphs (color classes) in each of which the degree of every vertex \( v \in V \) does not exceed \( f(v) \), where \( f : V \to \mathbb{N} \) is a function assigning a natural number \( f(v) \) to vertex \( v \in V \) \([8, 16]\). Figure 1(a) illustrates an \( f \)-coloring of a graph \( G \) with three colors, that is, a decomposition of \( G \) into three edge-disjoint subgraphs, which are indicated by solid, thick, and dashed lines. The simplest example of such a decomposition is an edge-coloring to color all edges of \( G \) so that no two adjacent edges are colored with the same color: An edge-coloring is an \( f \)-coloring where \( f : V \to \{1\} \) is a constant function. Other examples are the following \( f \)- and \([g,f]\)-factorizations \([2, 4, 12, 19]\). An \( f \)-factorization of \( G \) is a decomposition of \( G \) to edge-disjoint spanning subgraphs in each of which the degree of every vertex \( v \) is exactly equal to \( f(v) \). A \([g,f]\)-factorization of \( G \) is a decomposition of \( G \) to edge-disjoint subgraphs such that the degree \( d(v) \) of each vertex \( v \) satisfies \( g(v) \leq d(v) \leq f(v) \) for each subgraph, where \( g(v) \) is a nonnegative integer assigned to vertex...
The degree-constrained subgraphs decomposition problem is to find a decomposition into the minimum number of subgraphs. Since the edge-coloring problem is NP-complete [10], the degree-constrained subgraphs decomposition problem is also NP-complete in general. Therefore the theory of NP-completeness immediately implies that there exists a polynomial-time reduction of the degree-constrained subgraphs decomposition problem to the edge-coloring problem plausibly through another NP-complete problem, say 3-SAT [1]. However, no simple direct reduction has been known so far.

In this paper we show that the degree-constrained subgraphs decomposition problem can be simply reduced to the edge-coloring problem in polynomial time. We first give a very simple reduction of the $f$-coloring problem to the ordinary edge-coloring problem. That is, we show that, given a multigraph $G$ together with a function $f$ and a positive integer $k$, one can directly construct in polynomial-time a new simple graph $G_{f,k}$ such that there is an $f$-coloring of $G$ with at most $k$ colors if and only if there is an edge-coloring of $G_{f,k}$ with $k$ colors. It should be noted that the theory of NP-completeness does not imply the existence of such a single simple graph $G_{f,k}$. We construct $G_{f,k}$ from $G$ by inserting an appropriate bipartite graph $P(v)$ for each vertex $v \in V$, as illustrated in Fig. 1. We then derive, from the construction above, necessary and sufficient conditions for a graph $G$ to have an $f$-factorization or a $[g,f]$-factorization. Finding such conditions has been an open problem in graph theory [2]. The conditions immediately imply that the $f$- and $[g,f]$-factorization problems can be reduced to the edge-coloring problem in polynomial time. The construction also implies that the edge-coloring problem for multigraphs can be easily reduced to the edge-coloring problem for simple graphs. Thus we show that the degree-constrained subgraphs decomposition problem is not more intractable than the ordinary edge-coloring problem although the former
appears to be more difficult than the latter. An early version of the paper has been presented at [21].

2. PRELIMINARIES

In this section we first give some definitions, then review a trivial reduction, and finally observe that the \( f \)-coloring problem with \( k \) colors can be solved in linear time if \( k \leq 2 \).

Let \( G = (V, E) \) denote a graph with vertex set \( V \) and edge set \( E \). We often denote by \( V(G) \) and \( E(G) \) the vertex set and the edge set of \( G \), respectively. We assume that \( G \) has no selfloops but may have multiple edges, that is, \( G \) is a so-called multigraph. If \( G \) has no multiple edges, then \( G \) is called a simple graph. An edge joining vertices \( u \) and \( v \) is denoted by \((u, v)\). The degree of vertex \( v \in V(G) \) is denoted by \( d(v, G) \) or simply by \( d(v) \). The maximum degree of \( G \) is denoted by \( \Delta(G) \) or simply by \( \Delta \). A graph \( G = (V, E) \) is bipartite if \( V \) is partitioned to two partite sets \( U_b \) and \( W_b \) so that \( u \in U_b \) and \( v \in W_b \) for every edge \((u, v) \in E\).

An edge-coloring of a graph \( G = (V, E) \) is to color all the edges of \( G \) so that no two adjacent edges are colored with the same color. The minimum number of colors needed for an edge-coloring is called the chromatic index of \( G \) and denoted by \( \chi'(G) \). Clearly \( \chi'(G) \geq \Delta(G) \). Kőnig showed that \( \chi'(G) = \Delta(G) \) if \( G \) is bipartite [7, 14]. Vizing showed that \( \chi'(G) = \Delta(G) \) or \( \Delta(G) + 1 \) if \( G \) is a simple graph [7, 18]. The edge-coloring problem is to find an edge-coloring of \( G \) with \( \chi'(G) \) colors.

Let \( f : V \to \mathbb{N} \) be a function which assigns a natural number \( f(v) \in \mathbb{N} \) to each vertex \( v \in V \). We assume without loss of generality \( f(v) \leq d(v) \) for every vertex \( v \in V \). Then an \( f \)-coloring of \( G \) is to color all the edges of \( G \) so that, for each vertex \( v \in V \), at most \( f(v) \) edges incident to \( v \) are colored with the same color [8, 16]. Thus an \( f \)-coloring of \( G \) is a partition of \( E \) into subsets, each inducing a spanning subgraph whose vertex-degrees are bounded above by \( f \). An ordinary edge-coloring is a special case of an \( f \)-coloring such that \( f(v) = 1 \) for every vertex \( v \in V \). The minimum number of colors needed for an \( f \)-coloring is called the \( f \)-chromatic index of \( G \) and denoted by \( \chi'_f(G) \). The \( f \)-coloring problem is to find an \( f \)-coloring of \( G \) with \( \chi'_f(G) \) colors for a given graph \( G \). Let \( d_f(v, G) = \lceil d(v, G)/f(v) \rceil \) for \( v \in V \), and let \( \Delta_f(G) = \max \{ d_f(v, G) \mid v \in V(G) \} \). Clearly \( \chi'_f(G) \geq \Delta_f(G) \). It is known that \( \chi'_f(G) = \Delta_f(G) \) for any bipartite graph \( G \) and that \( \chi'_f(G) = \Delta_f + 1 \) for any simple graph \( G \) [8].

We have given the following trivial reduction of an \( f \)-coloring to an edge-coloring [20]. For each vertex \( v \in V \) of a graph \( G \), replace \( v \) with \( f(v) \) copies \( v_1, v_2, \ldots, v_{f(v)} \), and attach to the copies the \( d(v) \) edges which were incident to \( v \) in \( G \); attach \( \lceil d(v)/f(v) \rceil \) or \( \lfloor d(v)/f(v) \rfloor \) edges to each copy \( v_i \),
Let \( G_f \) be the resulting graph. It should be noted that the construction of \( G_f \) is not unique. Figure 2 illustrates \( G \) and an example of \( G_f \), where the number next to vertex \( v \) is \( f(v) \). Clearly \( \Delta(G_f) = \Delta_f(G) \). Since an edge-coloring of \( G_f \) immediately induces an \( f \)-coloring of \( G \) with the same number of colors, we have

\[
\chi'_f(G) \leq \chi'(G_f). \tag{1}
\]

If \( G \) is a simple graph then \( G_f \) is also a simple graph, and if \( G \) is bipartite then \( G_f \) is also bipartite. Therefore, the results of Vizing and König \cite{14, 18} together with the reduction above immediately imply that \( \chi'_f(G) = \Delta_f(G) \) or \( \Delta_f(G) + 1 \) if \( G \) is a simple graph and that \( \chi'_f(G) = \Delta_f(G) \) if \( G \) is bipartite. Thus the reduction is trivial but very useful. However, Eq. (1) does not always hold in equality. For example, \( \chi'_f(G) = 2 \) for a graph \( G \) in Fig. 2(a) as indicated by solid and dashed lines, but \( \chi'(G_f) = 3 \) for a graph \( G_f \) in Fig. 2(b) as indicated by solid, dashed and thick lines.

In the next section, for any positive integer \( k \) we give a sophisticated transformation of a multigraph \( G \) to a new simple graph \( G_{2k} \) such that \( \chi'_f(G) \leq k \) if and only if \( \chi'(G_{2k}) \leq k \). In particular for a simple graph \( G \), we show that Eq. (1) holds in equality: \( \chi'_f(G) = \chi'(G_{2k}) \) when \( k = \Delta_f(G) \).

Clearly a multigraph \( G \) satisfies \( \chi'_f(G) = 1 \) if \( \Delta_f(G) = 1 \). Furthermore one can easily observe the following lemma, whose proof will be given in Appendix A.

**Lemma 2.1.** Let \( G = (V, E) \) be a connected multigraph with \( \Delta_f(G) = 2 \). Then the following (a) and (b) hold:

(a) \( \chi'_f(G) \) is either 2 or 3; and

(b) \( \chi'_f(G) = 2 \) if and only if

- \( |E| \) is an even number, or
- \( d(v, G) \leq 2(v) - 1 \) for a vertex \( v \in V \).
Thus the $f$-coloring problem can be easily solved in linear time if $\Delta_f(G) \leq 2$. Therefore, in the remainder of this paper, we may assume that $\Delta_f(G) \geq 3$ and $k \geq 3$.

3. SIMPLE REDUCTION

In this section, for any $k \geq 3$ we give a direct and sophisticated transformation of a graph $G$ to a new graph $G_{fk}$ such that $\chi'_f(G) \leq k$ if and only if $\chi'(G_{fk}) \leq k$.

We use the following graph $P_{dk}$, called a $(d, k)$-permutation graph, as a building-block to construct $G_{fk}$ from $G$. (See Fig. 3.) For positive integers $d \geq 1$ and $k \geq 3$, let $P_{dk}$ be a bipartite simple graph such that:

- there are $d$ input vertices $U = \{u_1, u_2, \ldots, u_d\}$ and $d$ input edges $E_i = \{e_{i1}, e_{i2}, \ldots, e_{id}\}$ incident to input vertices;
- there are $d$ output vertices $W = \{w_1, w_2, \ldots, w_d\}$ and $d$ output edges $E_o = \{e_{o1}, e_{o2}, \ldots, e_{od}\}$ incident to output vertices;
- $d(\mathbf{v}, P_{dk}) = 1$ if $\mathbf{v} \in U \cup W$;
- otherwise.

Thus $\Delta(P_{dk}) = k$. Let $C = \{c_1, c_2, \ldots, c_k\}$ be any set of $k$ colors. We call $P_{dk}$ a $(d, k)$-permutation graph if the following Properties (i) and (ii) hold for edge-colorings of $P_{dk}$ with $k$ colors in $C$:

(i) the output color sequence is always a permutation of the input color sequence: more precisely, for any edge-coloring of $P_{dk}$ with $k$ colors, the sequence of colors of output edges is a permutation of that of input edges; and

(ii) the input color sequence can be arbitrarily permuted to the output color sequence: more precisely, for any sequence of colors $C_i = \{c_{i1}, c_{i2}, \ldots, c_{id}\}$ and for any permutation $C_o = \{c_{o1}, c_{o2}, \ldots, c_{od}\}$ of $C_i$,

![FIG. 3. Permutation graph $P_{dk}$.](image-url)
there is an edge-coloring of $P_{dk}$ with $k$ colors such that input edge $e_{ix}$ is colored by $c_{ix}$ and output edge $e_{ox}$ is colored by $c_{ox}$ for each $x$, $1 \leq x \leq d$.

The following lemma holds on $P_{dk}$.

**Lemma 3.1.** For any $d \geq 1$ and $k \geq 3$ there is a $(d, k)$-permutation graph $P_{dk}$ such that $|E(P_{dk})| = O(dk^3 \max\{1, \log_k d\})$.

A proof of Lemma 3.1 and a construction of $P_{dk}$ similar to the well-known Clos switching network [6, 11] will be given in Section 5.

We construct $G_{fk}$ from $G$ and copies of $P_{dk}$ as follows (see Fig. 1):

- (a) for each vertex $v \in V$, replace $v$ by a copy $P(v)$ of $P_{d(v)k}$, and merge the $d(v)$ output vertices $w_1, w_2, ..., w_{d(v)}$ of $P(v)$ to $f(v)$ vertices $v_1, v_2, ..., v_{f(v)}$ so that $d(v_j, G_{fk}) = \lceil d(v, G)/f(v) \rceil$ or $\lceil d(v, G)/f(v') \rceil$ for any $j$, $1 \leq j \leq f(v)$; and

- (b) for each edge $e = (v, v') \in E$, identify, as a single edge, an input edge of $P(v)$ and an input edge of $P(v')$ which are surrogates of $e$.

Clearly,

$A(G_{fk}) = \begin{cases} k, & \text{if } k \geq A_f(G); \\ A_f(G), & \text{otherwise}. \end{cases}$

Furthermore $G_{fk}$ is a simple graph even if $G$ has multiple edges. Figure 1(b) illustrates $G_{fk}$ for the graph $G$ in Fig. 1(a).

We have the following theorem on $G_{fk}$ as the main result of the paper.

**Theorem 3.2.** Let $G = (V, E)$ be any multigraph with $A_f(G) \geq 3$, and let $k$ be any integer with $k \geq 3$. Then

- (a) $\chi'_f(G) \leq k$ if and only if $\chi'(G_{fk}) = k$; and

- (b) the size of $G_{fk}$ is polynomial in $|E|$, more precisely

$|E(G_{fk})| = O(|E| A_f(G)^3 \log A_f A)$.

**Proof.** (a) For the case $k < A_f(G)$, we have $\chi'_f(G) > A_f(G) > k$, $\chi'(G_{fk}) > A_f(G) = A_f(G) = k$, and hence the claim holds. Thus we may assume that $k \geq A_f(G)$. In this case $A_f(G_{fk}) = k$.

Sufficiency. Suppose that $\chi'(G_{fk}) = k = A_f(G_{fk})$. Then there is an edge-coloring of $G_{fk}$ with $k$ colors, as illustrated in Fig. 1(b). For every vertex $v \in V$, all the edges incident to each of vertices $v_1, v_2, ..., v_{f(v)}$ are colored with different colors, and hence at most $f(v)$ of the $d(v)$ output edges of $P(v)$ are colored with the same color. Therefore, by Property (i) above, at
most $f(v)$ of the input edges are colored with the same color. Thus the coloring of input edges in $G_{f_{k}}$ immediately induces an $f$-coloring of $G$ with $k$ colors, as illustrated in Fig. 1(a). Hence $\chi'_f(G) \leq k$.

**Necessity.** Suppose that $\chi'_f(G) \leq k$. Then there is an $f$-coloring of $G$ with $k$ colors $c_1, c_2, \ldots, c_k$, as illustrated in Fig. 1(a). Construct a partial edge-coloring of $G_{f_{k}}$ in which every input edge $e_{ix}$ is assigned the same color as the edge of $G$ corresponding to $e_{ix}$. For every vertex $v \in V$, let $C_i = \{c_{i1}, c_{i2}, \ldots, c_{id(v)}\}$ be the input color sequence of $P(v)$ decided in this way. Since the same color appears in $C_i$ at most $f(v)$ times and $d(v)_{G_{f_{k}}} = \lceil d(v)/f(v) \rceil$ or $\lfloor d(v)/f(v) \rfloor$ for any $j$, $1 \leq j \leq f(v)$, one can easily observe that there is a permutation $C_o = \{c_{o1}, c_{o2}, \ldots, c_{od(v)}\}$ of $C_i$ such that all colors of the output edges incident to $v$ are distinct from each other for any $j$, $1 \leq j \leq f(v)$. Therefore, by Property (ii), one can extend the partial edge-coloring above to an edge-coloring of $G_{f_{k}}$ with $k$ colors, as illustrated in Fig. 1(b). Hence $\chi'(G_{f_{k}}) \leq k$. Since $\chi'(G_{f_{k}}) \geq \Delta(G_{f_{k}}) = k$, we have $\chi'(G_{f_{k}}) = k$.

(b) $|E(G_{f_{k}})|$ is polynomial in $|E|$.

Since $A_f(G) \leq \chi'_f(G) \leq 2A_f(G)$ (see Eq. (4) below), one may assume that $A_f(G) \leq k \leq 2A_f(G)$. Therefore by Lemma 3.1 we have

$$|E(P(v))| = O(d(v)(A_f)^3 \max \{1, \log_d A\})$$

for each vertex $v \in V$. Furthermore $d(v) \leq A(G)$ and $\sum_{v \in V} d(v) = 2|E|$. Thus we have

$$|E(G_{f_{k}})| \leq \sum_{v \in V} |E(P(v))| \leq O\left(\sum_{v \in V} d(v)(A_f)^3 \log_d A\right) \leq O(|E| (A_f)^3 \log_d A).$$

Q.E.D

The number of edges joining vertices $u$ and $v$ in a multigraph $G$ is called the **edge-multiplicity** of $(u, v)$, and denoted by $\mu(u, v)$. Let $\mu(u) = \max\{|\mu(u, v)| \mid (u, v) \in E\}$, and let $\mu(G) = \max\{|\mu(u)| \mid u \in V\}$. Let $\mu_f(G) = \max\{|\mu(v)/f(v)| \mid v \in V\}$, then $\mu_f(G) \leq A_f(G)$. Vizing [7, 18] showed that any multigraph $G$ satisfies

$$\chi'(G) \leq A(G) + \mu(G).$$

(2)
On the other hand, Hakimi and Kariv [8] showed that any multigraph $G$ satisfies

$$
\chi'_f(G) \leqslant \max_{v \in V} \left[ \frac{d(v) + \mu(v)}{f(v)} \right].
$$

This result immediately implies

$$
\chi'_f(G) \leq A_f(G) + \mu_f(G),
$$

and hence

$$
\chi'_f(G) \leq 2A_f(G).
$$

Eq. (3) can be immediately derived also from Eq. (2) and the trivial reduction illustrated in Fig. 2, because it is easy to construct $G_f$ such that $\Delta(G_f) = A_f(G)$ and $\mu(G_f) \leq \mu_f(G)$.

Theorem 3.2 and Eq. (3) immediately imply the following corollary.

**Corollary 3.3.** For any multigraph $G$ with $A_f(G) \geq 3$

$$
\chi'_f(G) = \min\{k \mid A_f(G) \leq k \leq A_f(G) + \mu_f(G), \chi'(G_{f_k}) = k\}.
$$

Given an edge-coloring of a simple graph $G_{f_k}$ with $\chi'(G_{f_k}) = k$ colors, one can immediately find an $f$-coloring of a multigraph $G$ with $k$ colors. Therefore the $f$-coloring problem for a multigraph is polynomial-time reducible to the ordinary edge-coloring problem for simple graphs. Indeed, by the binary search, one can solve the $f$-coloring problem for a multigraph $G$, solving the edge-coloring problem for simple graphs at most $\lceil \log_2(\mu_f(G) + 2) \rceil$ times.

The edge-coloring problem looks to be more intractable for multigraphs than for simple graphs, because $\chi'(G) = A$ or $A + 1$ for simple graphs $G$, but $A \leq \chi'(G) \leq A + \mu(G)$ for multigraphs $G$. However, since an edge-coloring is an $f$-coloring in which $f(v) = 1$ for each vertex $v \in V$, we have the following corollary.

**Corollary 3.4.** An edge-coloring problem for a multigraph can be easily reduced to an edge-coloring problem for simple graphs in polynomial-time.

We denote $G_{f_k}$ with $k = A_f(G)$ simply by $G_{f_k}$. Then $\Delta(G_{f_{k+1}}) = k = A_f(G)$, and Eq. (1) holds in equality for simple graphs as follows.
COROLLARY 3.5. For any simple graph $G$ with $A_f(G) \geq 3$

$$\chi'_f(G) = \chi'(G_{fA}).$$

Proof. We first claim $\chi'_f(G) \leq \chi'(G_{fA})$. Since $G$ and $G_{fA}$ are simple graphs, $\chi'(G_{fA}) = A(G_{fA})$ or $A(G_{fA}) + 1$, and $\chi'_f(G) = A_f(G)$ or $A_f(G) + 1$. If $\chi'_f(G_{fA}) = A(G_{fA}) + 1$, then

$$\chi'_f(G) \leq A_f(G) + 1 = A(G_{fA}) + 1 = \chi'(G_{fA}).$$

If $\chi'_f(G_{fA}) = A_f(G_{fA})$, then by Theorem 3.2 $\chi'_f(G) \leq A_f(G) = A(G_{fA}) = \chi'(G_{fA}).$

We next claim $\chi'_f(G) \geq \chi'(G_{fA})$. If $\chi'_f(G) = A_f(G) + 1$, then

$$\chi'_f(G) = A_f(G) + 1 = A(G_{fA}) + 1 \geq \chi'(G_{fA}).$$

If $\chi'_f(G) = A_f(G)$, then by Theorem 3.2 $\chi'_f(G) = A_f(G) = \chi'(G_{fA}).$ Q.E.D

4. FACTORIZATIONS

In this section, using the graph transformation in Section 3, we give necessary and sufficient conditions for a graph to have an $f$-factorization or a $[g,f]$-factorization.

An $f$-factor of a graph $G = (V,E)$ is a spanning subgraph $H$ of $G$ such that $d(v,H) = f(v)$ for every $v \in V$ \[9, 17\]. An $f$-factor in $G$ is drawn in thick lines in Fig. 4(a). Tutte showed that the existence of an $f$-factor in $G$ can be reduced to the existence of a 1-factor, that is, a perfect matching, in

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{tutte_transformation.png}
\caption{Tutte's transformation.}
\end{figure}
a new graphs $G_T$ constructed from $G$ as follows [17]. For each vertex $v$ of
$G$, replace $v$ with a complete bipartite graph $K_{d(v),d(v)-f(v)}$ and attach to
each of the $d(v)$ left vertices of $K_{d(v),d(v)-f(v)}$ one of the $d(v)$ edges which
were incident to $v$ in $G$. Let $G_T$ be the resulting graph. Then one can
observe that $G$ has an $f$-factor if and only if $G_T$ has a 1-factor. A 1-factor
in $G_T$ is drawn in thick lines in Fig. 4(b). (Tutte’s original construction
replaces each vertex $v$ of $G$ with a complete bipartite graph $K_{d(v),f(v)}$. In this
case, the complement of a 1-factor in a new graph corresponds to an
$f$-factor in $G$ [17].)

An $f$-factorization of a graph $G=(V,E)$ is a partition of set $E$ into subsets
each of which induces a $f$-factor [2]. An $f$-factorization into 1-factors is
called a 1-factorization. Thus an $f$-factorization of $G$ into $k$ factors is indeed
an $f$-coloring with $k$ colors in which each of the color classes induces an
$f$-factor. The trivial necessary condition for a multigraph $G$ to have an
$f$-factorization is that all vertices $v$ of $G$ satisfy

$$d(v) = kf(v)$$

for the same integer $k$. For the case $k = 1$, $G$ itself is an $f$-factorization of
$G$. For the case $k = 2$, Lemma 2.1 implies that $G$ has an $f$-factorization if
and only if every connected component of $G$ has an even number of edges.
For the case $k = 3$, we have the following corollary from Theorem 3.2.

**Corollary 4.1.** Let $G$ be a multigraph satisfying the trivial condition
Eq. (5) for $k \geq 3$. Then $G$ has an $f$-factorization if and only if $\varphi(G_{k'}) = k$,
that is, the $k$-regular graph $G_{k'}$ has a 1-factorization.

Corollary 4.1 has a resemblance to Tutte’s classical result above, and is
interesting in its own right since finding such a necessary and sufficient
condition for a graph to have an $f$-factorization has been an open problem
in graph theory [2].

Let $G=(V,E)$ be a graph, and let $f: V \rightarrow N$ and $g: V \rightarrow Z$ be functions
which assign to each vertex $v \in V$ a positive integer $f(v)$ and a nonnegative
integer $g(v)$ such that $g(v) \leq f(v)$. A $[g,f]$-factor of a graph $G=(V,E)$ is
a spanning subgraph $H$ of $G$ such that $g(v) \leq d(v,H) \leq f(v)$ for each vertex
$v \in V$ [3, 13]. A $[g,f]$-factorization of $G$ is a partition of set $E$ into subsets
each of which induces a $[g,f]$-factor [4, 12, 19]. The $f$-coloring is a special case of
the $[g,f]$-factorization in which $g: V \rightarrow \{0\}$ is a constant function.
The trivial necessary condition for a multigraph $G$ to have a
$[g,f]$-factorization into $k$ factors is that for every vertex $v$ in $G$

$$kg(v) \leq d(v) \leq kf(v).$$
The $[g, f]$-factorization problem is to find a $[g, f]$-factorization of a given graph $G$ into the minimum number of $[g, f]$-factors.

Cai\(^1\) has shown that the $[g, f]$-factorization problem can be reduced to the $f$-coloring problem, as follows [5]. Let $G$ be a multigraph satisfying the trivial necessary condition Eq. (6) for $k \geq 1$. For each vertex $v \in V$ such that $k(f(v) - d(v, G)) \geq 1$, add to $G$ a new vertex $v^*$ and $kf(v) - d(v, G)$ multiple edges joining $v$ and $v^*$. Let $G^*$ be the resulting graph. Define a new function $f^* : V(G^*) \to \mathbb{N}$ as follows

$$f^*(v) = f(v) \quad \text{if} \quad v \in V(G);$$

and

$$f^*(v^*) = f(v) - g(v) \quad \text{if} \quad v^* \in V(G^*) - V(G) \quad \text{and} \quad (v^*, v) \in E(G^*).$$

One can observe that $G$ has a $[g, f]$-factorization into $k$ factors if and only if $G^*$ can be $f^*$-colored with $k$ colors, that is, $\chi'_f(G^*) = k$ [5]. We therefore have the following corollary from Theorem 3.2.

**Corollary 4.2.** Let $G$ be a multigraph satisfying the trivial condition Eq. (6) for $k \geq 3$. Then a multigraph $G$ has a $[g, f]$-factorization into $k$ factors if and only if $\chi'(G^*_{sk}) = k$.

Thus the $f$- and $[g, f]$-factorization problems for a multigraph can be reduced to the ordinary edge-coloring problem for simple graphs in polynomial time.

**5. CONSTRUCTION OF PERMUTATION GRAPHS**

In this section we give a construction of a $(d, k)$-permutation graph $P_{d,k}$, and prove Lemma 3.1. The difficulty in the construction of $P_{d,k}$ stems from the degree requirement: all vertices $v$ except the input and output vertices must satisfy $d(v, P_{d,k}) = k$, regardless of the number $d$ of input vertices. We first recursively construct $P_{(k-1)^d,k}$, $q \in \mathbb{N}$, as illustrated in Fig. 5. We then construct $P_{d,k}$ from $P_{(k-1)^d,k}$, $d \leq (k-1)^q$, by identifying $(k-1)^q - d$ redundant pairs of input and output edges as single edges, as illustrated in Fig. 6.

For the case $q = 1$, we construct $P_{(k-1)^d,k}$ from a complete bipartite graph $K_{(k-1)^d-1, k-1}$ by attaching $k-1$ input edges to the left vertices and $k-1$ output edges to the right vertices of $K_{(k-1)^d-1, k-1}$. Thus $|E(P_{(k-1)^d,k})| = (k-1)(k+1)$, and $\lambda(P_{(k-1)^d,k}) = k$; all input and output vertices have degree 1 and all others have degree $k$.

\(^1\) The authors thank Professor M. C. Cai of the Chinese Academy of Science for helpful discussions on the $[g, f]$-factorization problem.
FIG. 5. Construction of a $((k - 1)^q, k)$-permutation graph with $q \geq 2$.

FIG. 6. Construction of $P_{dk}$ from $P_{(k-1)^qk}$.
For the case $q \geq 2$, we recursively construct $P_{(k-1)\varphi k}$ as illustrated in Fig. 5. The construction is similar to that of the well-known Clos switching network [6, 11, 15]. $P_{(k-1)\varphi k}$ consists of three stages connected in cascade. The first stage consists of $(k-1)^{\varphi -1}$ copies $B_{31}, B_{12}, ..., B_{(k-1)^{\varphi -1}}$ of $P_{(k-1)k}$. The second stage consists of $k-1$ copies $B_{32}, B_{22}, ..., B_{2(k-1)}$ of $P_{(k-1)^{\varphi -1}k}$. The third stage consists of $(k-1)^{\varphi -1}$ copies $B_{33}, B_{32}, ..., B_{(k-1)^{\varphi -1}}$ of $P_{(k-1)k}$. Denote by $e_{\alpha}(B)$ the $x$th input edge and by $e_{\alpha}(B)$ the $x$th output edge of a permutation graph $B$. For all integers $x$ and $y$, 1 $\leq x \leq k-1$ and $1 \leq y \leq (k-1)^{\varphi -1}$, the output edge $e_{\alpha}(B_{1y})$ is identified with the input edge $e_{\alpha}(B_{2x})$, and the input edge $e_{\alpha}(B_{3y})$ is identified with the output edge $e_{\alpha}(B_{2x})$.

One can easily known that $P_{(k-1)\varphi k}$ is a bipartite simple graph and $\Delta(P_{(k-1)\varphi k})=k$; all input and output vertices have degree 1, and all the others have degree $k$. It is known that such $P_{\varphi}$, $d=(k-1)^{\varphi}$, is a so-called switching network: For any permutation $\{j_1, j_2, ..., j_d\}$ of $\{1, 2, ..., d\}$, $P_{\varphi}$ contains $d$ vertex-disjoint paths, each starting at an input vertex $u_x \in U$ and ending at the output vertex $w_y \in W$, 1 $\leq x \leq d$ [6, 11]. We give a stronger result, that is, we prove that $P_{\varphi}$ satisfies Properties (i) and (ii), and hence have the following lemma.

**Lemma 5.1.** For any integers $k \geq 3$ and $q \geq 1$, there is a $((k-1)^{\varphi}, k)$-permutation graph $P_{(k-1)\varphi k}$ such that $|E(P_{(k-1)\varphi k})| < 2qk(k-1)^{\varphi}$.

**Proof.** (a) We first verify that $|E(P_{(k-1)\varphi k})| < 2qk(k-1)^{\varphi}$. The size of the graph $P_{(k-1)\varphi k}$ is

$$|E(P_{(k-1)\varphi k})| = (k-1) |E(P_{(k-1)^{\varphi -1}k})| + 2(k-1)^{\varphi -1} |E(P_{(k-1)k})|$$

$$- 2(k-1)^{\varphi}$$

$$= (k-1) |E(P_{(k-1)^{\varphi -1}k})| + 2k(k-1)^{\varphi}.$$  

Solving the recursive equation above, we have

$$|E(P_{(k-1)\varphi k})| = (2qk - k + 1)(k-1)^{\varphi} < 2qk(k-1)^{\varphi}.$$  

(b) We then prove that $P_{(k-1)\varphi k}$ satisfies Property (i). Let $U_{\varphi}$ and $W_{\varphi}$ be the partite sets of the bipartite graph $P_{(k-1)\varphi k}$, that is, let $P_{(k-1)\varphi k}=(U_{\varphi} \cup W_{\varphi}, E_{\varphi})$. Since $P_{(k-1)\varphi k}$ is symmetric, $|U_{\varphi}| = |W_{\varphi}|$. Furthermore one may assume that $U_{\varphi}$ contains all input vertices, and $W_{\varphi}$ contains all output vertices, that is, $U \subseteq U_{\varphi}$ and $W \subseteq W_{\varphi}$. Consider any edge-coloring of $P_{(k-1)\varphi k}$ with $k$ colors $c_1, c_2, ..., c_k$ in $C$. Exactly one of the $k$ edges incident to $v$ is colored by $c_j$ for every vertex $v \in (U_{\varphi} - U) \cup (W_{\varphi} - W)$ and every color $c_j \in C$. Therefore, if $n_j$ edges in $P_{(k-1)\varphi k}$ are colored by $c_j$, then $n_j - |U_{\varphi} - U|$ input edges and
Proof of Lemma 3.1. We now construct a required \((d, k)\)-permutation graph \(P_{dk}\) from \(P_{(k-1)^2k}\). (See Fig. 6.) We choose the first \(d\) input vertices \(u_1, u_2, \ldots, u_d\) of \(P_{(k-1)^2k}\) as the input vertices of \(P_{dk}\) and the first \(d\) output vertices \(w_1, w_2, \ldots, w_d\) of \(P_{(k-1)^2k}\) as the output vertices of \(P_{dk}\). Therefore \(q\) must satisfy \((k-1)^2d \geq q\). Thus we choose \(q = \max\{2, \lceil \log_{k-1} d \rceil\}\). Note that \(k \geq 3\) and \((k-1)^2d \leq (k-1)^2q\) for such \(q\). Then we have
\[
|E(P_{(k-1)^2k})| = O(qk(k-1)^2) = O(dk^3 \max\{1, \log_k d\}).
\]
Identify a redundant pair of input edge \(e_{ix}\) and output edge \(e_{ox}\) as a single edge for each \(x\), \(d+1 \leq x \leq (k-1)^2\). Then clearly the resulting graph is a \((d, k)\)-permutation graph, \(|E(P_{dk})| = O(dk^3 \max\{1, \log_k d\})\), and \(P_{dk}\) has no multiple edges since \(q \geq 2\). (See Figs. 5 and 6.) Thus we have proved Lemma 3.1. Q.E.D

6. CONCLUSION

In this paper we gave a sophisticated transformation of a multigraph \(G\) to a new simple graph \(G_{fk}\) such that \(G\) can be \(f\)-colored with at most \(k\) colors if and only if \(G_{fk}\) can be edge-colored with \(k\) colors, as illustrated in Fig. 1. Thus the \(f\)-coloring problem for a multigraph \(G\) can be directly reduced to an ordinary edge-coloring problem for a new simple graph \(G_{fk}\). The size of \(G_{fk}\) is polynomial in the size of \(G\), and one can transform \(G\) to \(G_{fk}\) sequentially in polynomial-time. It is easy to know that one can transform \(G\) to \(G_{fk}\) in parallel in \(O(\log |V|)\) time with a polynomial number of operations, and hence the \(f\)-coloring problem is NC-reducible to the edge-coloring problem [11]. Although we assumed for simplicity that \(G\) has no selfloops, our transformation works well even if \(G\) has selfloops.

An exact edge-coloring of \(G_{fk}\) with \(k = A(G_{fk})\) colors immediately yields an \(f\)-coloring of \(G\) with \(k\) colors. However, an approximate edge-coloring of the simple graph \(G_{fk}\) with \(A(G_{fk})+1\) colors does not yield any approximate \(f\)-coloring of \(G\) since Properties (i) and (ii) do not always hold for the edge-coloring of \(P_{dk}\) with \(k+1 = A(G_{fk})+1\) colors. Thus our reduction does not preserve the worst-case ratio.

We furthermore showed that \(\chi_f(G) = \chi_f(G_{fk})\) for simple graphs \(G\) and that various other degree-constrained subgraphs decomposition problems
such as an $f$-factorization and a $[g, f]$-factorization can also be reduced to an ordinary edge-coloring problem.

Our transformation has a resemblance to Tutte's classical one to reduce the existence of an $f$-factor in a graph $G$ to that of a perfect matching in a new graph $G_T$ as illustrated in Fig. 4; both transformations use bipartite graphs as building blocks to construct a new graph, and reduce the general problems to their simplest versions. However, our building blocks are complicated ones constructed from Clos' switching networks as in Figs. 5 and 6 although Tutte's building blocks are simply complete bipartite graphs. Furthermore the idea behind our construction is different from Tutte's: Our building blocks permute color sequences, while in his transformation, the vertices in a new graph $G_T$ that are not covered by the edges corresponding to an $f$-factor in $G$ are covered by matchings in complete bipartite graphs.

APPENDICES

Appendix A: Proof of Lemma 2.1.

(a) Since $2 = A_f(G) \leq \chi'_f(G) \leq 3$. If $G$ has a vertex of odd degree, then add a new vertex $u$ to $G$ and join $u$ to each vertex of odd degree. Let $G'$ be the resulting graph. Note that $G = G'$ if $G$ has no vertex of odd degree in $G$. Since $G'$ is connected and all vertices in $G'$ have even degrees, $G'$ is an Eulerian graph and has an Eulerian circuit. Color all edges of $G'$ alternatively with colors $c_1$ and $c_2$ along the Eulerian circuit except the last edge, and color the last edge with color $c_3$. Then the coloring restricted to the edges of $G$ is an $f$-coloring of $G$ with at most three colors, because $A_f(G) = 2$ and hence $\lfloor d(v, G)/2 \rfloor \leq f(v)$ and at most $\lfloor d(v, G)/2 \rfloor$ edges incident to vertex $v$ are colored by the same color for every vertex $v \in V$ even if $v$ is the starting vertex of the circuit and $|E|$ is odd. Thus $\chi'_f(G) \leq 3$.

(b) We first prove the necessity. Suppose for a contradiction that $\chi'_f(G) = 2$ but $|E|$ is odd and $d(v, G) = 2f(v)$ for every vertex $v \in V$. Then, in an $f$-coloring of $G$ with two colors $c_1$ and $c_2$, both the number of edges colored by $c_1$ and that by $c_2$ are equal to $(\sum_{v \in V} f(v))/2$, and hence $|E|$ must be even, a contradiction.

We next prove the sufficiency: if either $|E|$ is even or $d(v, G) \leq 2f(v) - 1$ for a vertex $v$, then $\chi'_f(G) = 2$. There are the following three cases.

Case 1. $G$ is not an Eulerian graph.

In this case $G$ has a vertex of odd degree. Add a new vertex $u$ to $G$ and join $u$ to each vertex of odd degree. Let $G'$ be the resulting Eulerian graph.
Color the edges of $G$ alternatively with $c_1$ and $c_2$ along an Eulerian circuit starting and ending at $u$. Then the coloring restricted to the edges of $G$ is an $f$-coloring of $G$ with two colors $c_1$ and $c_2$ since $\lceil d(v)/2 \rceil \leq f(v)$ and at most $\lceil d(v)/2 \rceil$ edges incident to $v$ are colored by the same color for every vertex $v \in V$.

**Case 2.** $G$ is an Eulerian graph and $d(w, G) \leq 2f(w) - 2$ for a vertex $w \in V$.

Color the edges of $G$ alternatively with $c_1$ and $c_2$ along an Eulerian circuit starting and ending at $w$. Even if $|E|$ is odd, at most $\frac{1}{2}d(w, G) + 1 (\leq f(w))$ edges incident to $w$ are colored by the same color. Therefore the coloring is an $f$-coloring of $G$.

**Case 3.** $G$ is an Eulerian graph, $d(v, G) = 2f(v)$ for every vertex $v \in V$, and $|E|$ is even.

Color the edges of $G$ alternatively with colors $c_1$ and $c_2$ along any Eulerian circuit of $G$. Since $|E|$ is even, the coloring is an $f$-coloring of $G$ with two colors $c_1$ and $c_2$.

**Q.E.D.**

**Appendix B: Proof for (ii)**

We prove by induction on $q$ that $P_{(k-1)q}$ satisfies Property (ii).

As the induction base, we first prove that $P_{(k-1)q}$ satisfies Property (ii), that is, for any color sequence $C_i = \{c_{i1}, c_{i2}, ..., c_{i(k-1)}\}$ and any permutation $C_o = \{c_o1, c_o2, ..., c_{o(k-1)}\}$ of $C_i$, $P_{(k-1)q}$ can be edge-colored with $k$ colors so that input edge $e_{ia}$ is colored by $c_{ia}$ and output edge $e_{oa}$ is colored by $c_{oa}$ for each $x_1 \leq x \leq k-1$. By Property (i) the output color sequence is a permutation of the input color sequence for any edge-coloring of $P_{(k-1)q}$ with $k$ colors. Furthermore both the set of $k-1$ input vertices and the set of $k-1$ output vertices are symmetric each other in graph $P_{(k-1)q}$ since it is constructed from a complete bipartite graph $K_{k-1,k-1}$. Therefore it suffices to show that $P_{(k-1)q}$ can be edge-colored with $k$ colors so that the input color sequence is a permutation of $C_i$. Since $C_i$ contains at most $k-1$ colors, one of the $k$ colors, say $c \in C_i$, does not appear in $C_i$. Color $K_{k-1,k-1}$ with the $k-1$ colors in $C - \{c\}$, and color all the input and output edges of $P_{(k-1)q}$ with the color $c$. Let $c'$ be a color appearing in $C_i, n'$ (\( \geq 1 \)) times. Then $P_{(k-1)q}$ contains at least $n'$ vertex-disjoint $cc'$-alternating paths whose edges are colored alternately by $c$ and $c'$. Each of these paths starts at an input edge colored by $c$, and ends at an output edge colored by $c$. Choose arbitrarily exactly $n'$ paths among these $cc'$-alternating paths, and interchange the colors $c$ and $c'$ on the $n'$ paths. Then exactly $n'$ input edges and $n'$ output edges are colored by $c'$ in the resulting coloring of $P_{(k-1)q}$ with $k$ colors. Repeating the operation above for all other colors in $C_i$ one can obtain an edge-coloring of $P_{(k-1)q}$ with $k$ colors such that
the input color sequence is a permutation of \(C_i\). Thus we have verified that \(P_{(k-1)q}^{(k-1)q} \) satisfies Property (ii).

We next assume as the induction hypothesis that \(P_{(k-1)q}^{(k-1)q} \) satisfies Property (ii), and prove that \(P_{(k-1)q}^{(k-1)q} \) satisfies Property (ii): The output color sequence can be an arbitrary permutation \(C_o = \{c_{o1}, c_{o2}, \ldots, c_{o(k-1)q}\}\). We first color the input edge \(e_{ix}\) by \(c_{ix}\) and the output edge \(e_{ox}\) by \(c_{ox}\) for each \(x\), \(1 \leq x \leq (k-1)^q\). We then decide the colors of edges in blocks on the second stage from the following bipartite multigraph \(G_B = (U_B \cup W_B, E_B)\) constructed from \(C_i\) and its arbitrary permutation \(C_o\). The partite sets \(U_B\) and \(W_B\) consist of the blocks on the first and the third stages, respectively:

\[
U_B = \{B_{11}, B_{12}, \ldots, B_{1(k-1)^{q-1}}\};
\]

and

\[
W_B = \{B_{31}, B_{32}, \ldots, B_{3(k-1)^{q-1}}\}.
\]

If the arbitrary permutation above permutes an input color \(c_{ix}\) to an output color \(c_{oy}\), then we join vertices \(B_{1j}\) and \(B_{3l}\) in graph \(G_B\) where \(B_{1j}\) is the block to which the input edge \(e_{ix}\) is connected and \(B_{3l}\) is the block to which the output edge \(e_{oy}\) is connected. Thus

\[
E_B = \{(B_{1j}(x), B_{3l}(x)) \mid 1 \leq x \leq (k-1)^q, \text{ color } c_{ix} \text{ is permuted to color } c_{oy}, \}
\]

\[
\begin{align*}
(j(x) &= \lceil x/(k-1) \rceil, \text{ and } \\
(l(x) &= \lceil y/(k-1) \rceil).
\end{align*}
\]

Since every vertex of the bipartite graph \(G_B\) has degree \(k-1\), \(G_B\) can be edge-colored with \(k-1\) colors. Therefore \(E_B\) can be partitioned into \(k-1\) perfect matchings \(M_1, M_2, \ldots, M_{k-1}\) of \(G_B\), each of which contains \((k-1)^{q-1}\) edges. From \(M_1\) we decide the colors of input and output edges of \(B_{21}\) as follows. For each of the \((k-1)^{q-1}\) edges \((B_{1j}(x), B_{3l}(x))\) of \(M_1\), we color both the first output edge of block \(B_{1j}(x)\) and the first input edge of block \(B_{3l}(x)\) with color \(c_{ix} = c_{oy}\). Then by the construction of \(P^{(k-1)q}\) these edges are input and output edges of \(B_{21}\). (See Fig. 5.) Furthermore the output color sequence of \(B_{21}\) decided in this way is a permutation of the input color sequence of \(B_{21}\). Therefore, by the inductive hypothesis, such a coloring of input and output edges of \(B_{21}\) can be extended to an edge-coloring of \(B_{21}\) with \(k\) colors. Similarly we decide the edge-colorings of \(B_{22}, B_{23}, \ldots, B_{2(k-1)}\) from perfect matchings \(M_2, M_3, \ldots, M_{k-1}\), respectively. We finally decide the coloring of blocks on the first and third stages. Clearly the output color sequence is a permutation of the input color sequence.
for each of blocks \( B_{i1}, B_{i2}, \ldots, B_{(k-1)q}, \ldots, B_{31}, B_{32}, \ldots, B_{3(k-1)q} \). Since \( P_{(k-1)q} \) satisfies Property (ii), these color sequences can be extended to edge-colorings of these blocks with \( k \) colors. Thus we have completed an edge-coloring of \( P_{(k-1)q} \) with \( k \) colors such that the input color sequence is \( C_i \) and the output color sequence is the specified permutation \( C_o \) of \( C_i \).

Q.E.D

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