A LINEAR-TIME ALGORITHM
TO FIND FOUR INDEPENDENT SPANNING TREES
IN FOUR CONNECTED PLANAR GRAPHS

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ABSTRACT

Given a graph \( G \), a designated vertex \( r \) and a natural number \( k \), we wish to find \( k \) "independent" spanning trees of \( G \) rooted at \( r \), that is, \( k \) spanning trees such that the \( k \) paths connecting \( r \) and any vertex \( v \) in the \( k \) trees are internally disjoint. In this paper we give a linear-time algorithm to find four independent spanning trees in a 4-connected planar graph.

Keywords: Graph, independent spanning tree, algorithm.

1. Introduction

Given a graph \( G = (V, E) \), a designated vertex \( r \in V \) and a natural number \( k \), we wish to find \( k \) spanning trees \( T_1, T_2, \cdots, T_k \) of \( G \) such that, for any vertex \( v \), the \( k \) paths connecting \( r \) and \( v \) in \( T_1, T_2, \cdots, T_k \) are internally disjoint, that is, any two of them have no common intermediate vertices. Such \( k \) trees are called independent spanning trees of \( G \) (rooted at \( r \)). Four independent spanning trees are drawn by thick lines in Fig. 1. Note that independent spanning trees are not always edge-disjoint. Independent spanning trees have applications to fault-tolerant protocols in networks [1, 5, 8, 14].

It is conjectured that, for any \( k \geq 1 \), every \( k \)-connected graph \( G \) has \( k \) independent spanning trees rooted at any vertex [10, 15]. If \( G \) is a biconnected graph, then one can find two independent spanning trees in linear time by means of the st-numbering [2, 8]. If \( G \) is triconnected graph, then one can find three independent

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Fig. 1. Four independent spanning trees $T_1, T_2, T_3$ and $T_4$ of a graph $G$ rooted at $r$. 
spanning trees in $O(n^2)$ time by means of the ear-decomposition, where $n$ is the number of vertices of $G$ [2, 4]. If $G$ is a triconnected planar graph, then one can find three independent spanning trees in linear time by means of the canonical ordering [2]. On the other hand, it is not known whether the conjecture holds for $k \geq 4$ in general. However, Huck has recently shown that the conjecture holds for $k = 4$ if $G$ is planar, proving that every 4-connected planar graph has four independent spanning trees [7]. The proof in [7] yields an algorithm to actually find four independent spanning trees, but it takes time $O(n^3)$.

In this paper we give a simple linear-time algorithm to find four independent spanning trees of a 4-connected planar graph. Our algorithm is based on the “4-canonical decomposition” of a 4-connected planar graph [13], which is a generalization of the $st$-numbering [6], the canonical ordering [3] and the canonical 4-ordering [9].

The remainder of the paper is organized as follows. In Section 2 we introduce some definitions. In Section 3 we present our algorithm to find four independent spanning trees. In Section 4 we verify the correctness of our algorithm. Finally we conclude in Section 5 with some general comments and an open problem. An early version of the paper was presented in [11].

2. Preliminaries

In this section we introduce some definitions.

Throughout the paper, we consider a simple connected graph $G$. A graph $G$ with vertex set $V$ and edge set $E$ is denoted by $G = (V, E)$. We denote by $n$ the number of vertices in $G$, and assume that $n \geq 4$. An edge joining vertices $u$ and $v$ is denoted by $(u, v)$. The degree of a vertex $v$ in $G$ is the number of neighbors of $v$ in $G$ and is denoted by $d(v, G)$ or simply by $d(v)$. The connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph $K_1$. A graph $G$ is $k$-connected if $\kappa(G) \geq k$. A path passing through vertices $v_1, v_2, \ldots, v_l$ in this order is denoted by the sequence $v_1, v_2, \ldots, v_l$, where $(v_i, v_{i+1})$ is an edge for every $i$, $1 \leq i \leq l - 1$. We say that a path is simple if all the vertices are distinct from each other. We also say that a set of paths having common start and end vertices are internally disjoint if any pair of paths in the set have no common intermediate vertex. A cycle is a path beginning and ending with the same vertex. A cycle $v_1, v_2, \ldots, v_l (= v_1)$ is simple if all vertices $v_1, v_2, \ldots, v_{l-1}$ and all edges are distinct with each other.

A graph is planar if it can be embedded on the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A plane graph is a planar graph with a fixed plane embedding. The contour $C_o(G)$ of a plane connected graph $G$ is the cycle on the outer face. We always regard $C_o(G)$ as a clockwise cycle. Thus we write $C_o(G) = w_1, w_2, \ldots, w_l$ if the vertices $w_1, w_2, \ldots, w_l$ on $C_o(G)$ appear clockwise in this order, where $w_l = w_1$. If $G$ is biconnected, then $C_o(G)$ is a simple cycle.
3. Algorithm

In this section we give our algorithm to find four independent spanning trees of a 4-connected planar graph \( G \) rooted at any designated vertex \( r \).

Our main idea is very simple as follows. For each vertex \( v \), we assign four edges incident to \( v \) as the right leg \( rl(v) \), left leg \( ll(v) \), left arm \( la(v) \), and right arm \( ra(v) \) of \( v \); they appear clockwise around \( v \) in this order as illustrated in Fig. 2. Intuitively, the right leg \( rl(v) \) is the edge joining \( v \) and the “lower rightmost neighbor” of \( v \) in a plane embedding of \( G \) having the root \( r \) at the top. Similarly, the left leg \( ll(v) \) is the edge joining \( v \) and the “lower leftmost neighbor,” the left arm \( la(v) \) is the edge joining \( v \) and the “upper leftmost neighbor,” the right arm \( ra(v) \) is the edge joining \( v \) and the “upper rightmost neighbor.” The four subgraphs of \( G \), each of which is induced by all right legs, all left legs, all left arms, or all right arms, are four independent spanning trees of \( G \), as seen later. The end of right leg \( rl(v) \) other than \( v \) is called the right foot \( rf(v) \) of \( v \). Similarly the left foot \( lf(v) \), the left hand \( lh(v) \) and the right hand \( rh(v) \) are defined. The four neighbors \( rf(v), lf(v), lh(v) \) and \( rh(v) \) of \( v \) appear clockwise around \( v \) in this order.

![Fig. 2. Legs, arms, feet, and hands of a vertex \( v \).](image)

The assignment above can be done by means of the “4-canonical decomposition” of a graph, which is defined as follows. Given a 4-connected planar graph \( G = (V, E) \) and a vertex \( r \in V \) designated as a root, we first find a plane embedding of \( G \) in which \( r \) is located on the top of \( C_0(G) \) [12]. Let \( r_1, r_2, \cdots, r_{d(r)} \) be the neighbors of \( r \) in \( G \). Since \( G \) is 4-connected, \( d(r) \geq 4 \). Let \( G' \) be the plane graph obtained from \( G \) by deleting root \( r \) and all the edges \((r, r_1), (r, r_2), \cdots, (r, r_{d(r)})\) incident to \( r \). Fig. 3 (a) depicts \( G \), and Fig. 3 (b) depicts \( G' \). One may assume that \( r_1, r_2, \cdots, r_{d(r)} \) appear clockwise on \( C_0(G') \) in this order. We add to \( G' \) two new
Fig. 3. (a) Four-connected plane graph $G$, (b) plane graph $G'$, (c) plane graph $G''$, (d) four leg and arm paths starting from a vertex $v$ in $G''$, and (e) four internally disjoint paths between $r$ and $v$ in $G$. 
vertices, the bottom root $r_b$ and the top root $r_t$. We then join $r_b$ with $r_1$ and $r_2$, and join $r_t$ with $r_3, r_4, \ldots, r_{d(r)}$, as illustrated in Fig 3 (c). Let $G'' = (V'' , E'')$ be the resulting plane graph. Then $V'' = V \cup \{r_b, r_t\} - \{r\}$, and vertices $r_1, r_2, r_3, r_t$ and $r_{d(r)}$ appear on $C_o(G'')$ clockwise in this order. Let $m$ be a natural number, and let $\Pi = (W_1, W_2, \ldots, W_m)$ be a partition of set $V''$ to $m$ subsets $W_1, W_2, \ldots, W_m$ of $V''$, where $W_1 = \{r_b\}$ and $W_m = \{r_t\}$. In Fig. 3 (c) each set $W_i$ is indicated by an oval drawn in a dotted line. We say that the rank of a vertex $v \in V''$, denoted by $\text{rank}(v)$, is $i$ if $v \in W_i$ for $i, 1 \leq i \leq m$. We denote by $G_i, 1 \leq i \leq m$, the plane subgraph of $G''$ induced by all vertices of rank $\leq i$. We denote by $\overline{G_{i-1}}, 1 \leq i \leq m$, the plane subgraph of $G''$ induced by all vertices of rank $\geq i$. In Fig. 4, $G_i$ is darkly shaded, while $\overline{G_{i-1}}$ is lightly shaded. We assume that both $G_i$ and $\overline{G_{i-1}}$ are biconnected for each $i, 2 \leq i \leq m - 1$. We furthermore assume that if $W_i = \{u_1, u_2, \ldots, u_h\}$ then all vertices $u_1, u_2, \ldots, u_h$ in $W_i$ consecutively appear clockwise on $C_o(G_i)$ in this order for every $i, 1 \leq i \leq m$. Thus, for each $i, 2 \leq i \leq m$, all vertices in $W_i$ are located outside $C_o(G_{i-1})$ in the plane graph $G_i$. Then the partition $\Pi$ is called a $4$-canonical decomposition of $G$ (rooted at $r$) if, for each $i, 2 \leq i \leq m - 1$, one of the following three conditions holds:

(a) $W_i$ consists of exactly one vertex having at least two neighbors of rank $> i$ and at least two neighbors of rank $< i$. (See Fig. 4 (a).)

(b) $W_i$ consists of two or more vertices; every vertex in $W_i$ has at least two neighbors of rank $> i$; each of the first and the last vertices in $W_i$ has exactly one neighbor of rank $i$ and exactly one neighbor of rank $< i$; and each of the other vertices in $W_i$ has exactly two neighbors of rank $i$, and has no neighbor of rank $< i$. (See Fig. 4 (b).)
(c) $W_i$ consists of two or more vertices; every vertex in $W_i$ has at least two neighbors of rank $< i$; each of the first and the last vertices in $W_i$ has exactly one neighbor of rank $i$ and exactly one neighbor of rank $> i$; and each of the other vertices in $W_i$ has exactly two neighbors of rank $i$, and has no neighbor of rank $> i$. (See Fig. 4 (c).)

Note that conditions (b) and (c) are symmetric. Fig. 3 (c) illustrates a 4-canonical decomposition $\Pi = (W_1, W_2, \ldots, W_{12})$ of the graph $G$ in Fig. 3 (a). Subsets $W_4, W_5, W_6$ and $W_{10}$ satisfy condition (a), $W_2, W_3$ and $W_7$ satisfy condition (b), and $W_5, W_9$ and $W_{11}$ satisfy condition (c).

The vertex of rank 1, i.e. the bottom root $r_b$, has exactly two neighbors of larger ranks in $G''$, while the vertex of rank $m$, i.e. the top root $r_t$, has at least two neighbors of smaller ranks. Each vertex $v$ of rank $i$, $2 \leq i \leq m - 1$, has at least two neighbors of ranks $\geq i$ and has at least two neighbors of ranks $\leq i$ whichever condition (a), (b) or (c) holds for $i$; and furthermore each vertex $v$ of rank $i$ has at least two neighbors of rank $i$, and if $v$ has exactly two neighbors of rank $i$ then the ranks of all the other neighbors of $v$ are either greater than $i$ or smaller than $i$.

The 4-canonical decomposition is a generalization of the “st-numbering” [6], the “canonical decomposition” [3] and the “canonical 4-ordering” [9]. Although the definition of a 4-canonical decomposition above is slightly different from that in [13], they are effectively equivalent with each other and we have the following lemma.

**Lemma 1** Let $G = (V, E)$ be a 4-connected plane graph, and let $r$ be a designated vertex on $C_o(G)$. Then $G$ has a 4-canonical decomposition $\Pi$ rooted at $r$. Furthermore $\Pi$ can be found in linear time.

**Proof.** Similar to the proof of Lemma 3 in [13].

Let $v$ be any vertex in $G''$ other than $r_b$ and $r_t$. We write $N(v) = \{v_1, v_2, \ldots, v_{d(v,G'')}\}$ if the neighbors $v_1, v_2, \ldots, v_{d(v,G'')} \ (v \in G'')$ of $v$ in $G''$ appear clockwise around $v$ in this order. All $v$'s neighbors having ranks $\geq \text{rank}(v)$ consecutively appear around $v$, and so do all neighbors having ranks $\leq \text{rank}(v)$. This is the reason why one can define legs and arms of $v$ as follows. We assign four distinct edges incident to $v$ in $G''$ as the right leg $rl(v)$, the left leg $ll(v)$, the left arm $la(v)$ and the right arm $ra(v)$ of $v$ in a way that the following four statements (S1)–(S4) hold. (See Fig. 2.)

(S1) The two feet of $v$ have ranks $\leq \text{rank}(v)$.

(S2) The two hands of $v$ have ranks $\geq \text{rank}(v)$.

(S3) The right arm, the right leg, the left leg and the left arm of $v$ appear clockwise around $v$ in this order; in particular, the right arm precedes the right leg, and the left leg precedes the left arm.

(S4) For any foot $u$ and any hand $w$ of $v$, $\text{rank}(u) < \text{rank}(w)$.
For each vertex \( v \in W_i, 2 \leq i \leq m - 1 \), the definition above assigns four edges as legs and arms of \( v \), as follows.

**Case 1:** \( W_i \) satisfies condition (a). (See Fig. 5.)

In this case \( W_i = \{ u \} \) for a vertex \( u \in V'' \). Let \( v_1 \) be the vertex preceding \( u \) on \( C_o(G_i) \), and let \( N(u) = \{ v_1, v_2, \ldots, v_d(u) \} \). Then \( ll(u) = (u, v_1) \) and \( la(u) = (u, v_2) \). Let \( v_s, 1 \leq s \leq d(u) \), be the vertex succeeding \( u \) on \( C_o(G_i) \), then \( ra(u) = (u, v_{s-1}) \) and \( rl(u) = (u, v_s) \).

**Case 2:** \( W_i \) satisfies condition (b). (See Fig. 6.)

Let \( W_i = \{ u_1, u_2, \ldots, u_h \} \), where \( u_1, u_2, \ldots, u_h \) consecutively appear on \( C_o(G_i) \) clockwise in this order. Let \( u_0 \) be the vertex preceding \( u_1 \) on \( C_o(G_i) \), and let \( u_{h+1} \) be the vertex succeeding \( u_h \) on \( C_o(G_i) \). Then, for each vertex \( u_j \in W_i \), \( ll(u_j) = (u_j, u_{j-1}) \) and \( rl(u_j) = (u_j, u_{j+1}) \). Let \( N(u_j) = \{ v_1, v_2, \ldots, v_d(u_j) \} \), \( v_1 = u_{j-1} \) and \( v_d(u_j) = u_{j+1} \), then \( la(u_j) = (u_j, v_2) \) and \( ra(u_j) = (u_j, v_{d(u_j)-1}) \).

**Case 3:** \( W_i \) satisfies condition (c). (See Fig. 7.)

Let \( W_i = \{ u_1, u_2, \ldots, u_h \} \), where \( u_1, u_2, \ldots, u_h \) consecutively appear on \( C_o(G_i) \) clockwise in this order. Let \( u_0 \) be the vertex succeeding \( u_1 \) on \( C_o(G_{i-1}) \), and let \( u_{h+1} \) be the vertex preceding \( u_h \) on \( C_o(G_{i-1}) \). Then, for each vertex \( u_j \in W_i \), \( ra(u_j) = (u_j, u_{j+1}) \) and \( la(u_j) = (u_j, u_{j-1}) \). Let \( N(u_j) = \{ v_1, v_2, \ldots, v_d(u_j) \} \), \( v_1 = u_{j+1} \), and \( v_d(u_j) = u_{j-1} \), then \( rl(u_j) = (u_j, v_2) \) and \( ll(u_j) = (u_j, v_{d(u_j)-1}) \).

![Fig. 5. Assignment for Case 1.](image)

For each vertex \( v \in V'' - \{ r_b, r_i \} \), the **right leg path** \( P_{rl}(v) = v_1, v_2, \ldots, v_j \) is defined to be the path connecting \( v = v_1 \) and \( r_b = v_j \) such that \( rl(v_i) = (u_i, u_{i+1}) \) for every \( i, 1 \leq i \leq j - 1 \). Similarly, for each \( v \), we define the **left leg path** \( P_{ll}(v) \) connecting \( v \) and \( r_b \), the **left arm path** \( P_{la}(v) \) connecting \( v \) and \( r_i \), the **right arm path** \( P_{ra}(v) \) connecting \( v \) and \( r_i \). Intuitively, the right leg path goes "southeast," the left
Fig. 6. Assignment for Case 2.

Fig. 7. Assignment for Case 3.
leg path goes "southwest," the left arm path goes "northwest," and the right arm path goes "northeast" if \( r_t \) and \( r_b \) are located at the top and bottom, respectively, in the plane graph \( G'' \). In Fig. 3 (d) the four paths starting from a vertex \( v \) in \( G'' \) are drawn by thick lines, and in Fig. 3 (e) the corresponding four internally disjoint paths between \( r \) and \( v \) in \( G \) are drawn by thick lines.

We have the following lemma.

**Lemma 2** \( P_{rl}(v) \) and \( P_{lt}(v) \) are simple paths connecting \( v \) and \( r_b \) in \( G'' \), and \( P_{la}(v) \) and \( P_{ra}(v) \) are simple paths connecting \( v \) and \( r_t \) in \( G'' \).

**Proof.** We prove only that \( P_{rl}(v) \) is a simple path connecting \( v \) and \( r_b \) in \( G'' \). Proofs for the other paths are similar. Let \( P_{rl}(v) = v_1, v_2, \ldots, v_j \) where \( v_1 = v \). The property (S1) of legs implies that \( \text{rank}(v_1), \text{rank}(v_2), \ldots, \text{rank}(v_j) \) are non-increasing, that is

\[ \text{rank}(v_1) \geq \text{rank}(v_2) \geq \cdots \geq \text{rank}(v_j). \]

We first show that \( P_{rl}(v) \) is a simple path, that is, any two vertices \( v_i \) and \( v_{i'} \) in \( P_{rl}(v) \) are distinct. One may assume that \( 1 \leq i < i' \leq j \). Let \( k = \text{rank}(v_i) \) and \( k' = \text{rank}(v_{i'}) \), then \( k \geq k' \). If \( k > k' \), then obviously \( v_i \) and \( v_{i'} \) are distinct. Thus one may assume that \( k = k' \). Then \( v_i, v_{i'} \in W_k \), and hence \( W_k \) satisfies condition (b); otherwise, \( W_k \) would satisfy condition (a) or (c), and hence \( |W_k| = 1 \). (See Fig. 7.) Since \( W_k \) satisfies condition (b), the subpath of \( P_{rl}(v) \) between \( v_i \) and \( v_{i'} \) is a simple path all vertices of which have rank \( k \). (See Fig. 6.) Therefore \( v_i \) and \( v_{i'} \) are distinct.

We next show that \( P_{rl}(v) \) ends at \( r_b \). Every vertex in \( G'' \) except \( r_b \) and \( r_t \) has a right leg. Vertex \( r_b \) is a right foot of \( r_1 \), but \( r_t \) is not a right foot of any vertex in \( G'' \). Furthermore \( P_{rl}(v) \) is a finite simple path. Therefore \( P_{rl}(v) \) must end at \( r_b \). \( \square \)

As seen later in Lemmas 5 and 7, these four paths starting from \( v \) have no common intermediate vertex, and hence these four paths are internally disjoint if the bottom root \( r_b \) and the top root \( r_t \) in \( G'' \) are regarded as the root \( r \) in \( G \).

We are now ready to give our algorithm.

**Procedure FourTrees**(\( G, r \))

begin
1. Find a plane embedding of \( G \) such that \( r \in C_b(G) \);
2. Find a 4-canonical decomposition \( \Pi = (W_1, W_2, \ldots, W_m) \) of \( G \) rooted at \( r \);
3. Find \( rl(v), ll(v), la(v) \) and \( ra(v) \) for each vertex \( v \in V'' - \{r_b, r_t\} \);
4. Let \( T_{rl}, T_{lt}, T_{la} \) and \( T_{ra} \) be subgraphs of \( G'' \) induced by the right legs, left legs, left arms and right arms of all vertices in \( V'' - \{r_b, r_t\} \), respectively;
5. Regard vertex \( r_b \) in trees \( T_{rl} \) and \( T_{lt} \) as vertex \( r \);
6. Regard vertex \( r_t \) in trees \( T_{la} \) and \( T_{ra} \) as vertex \( r \);
7. return \( T_{rl}, T_{lt}, T_{la} \) and \( T_{ra} \) as four independent spanning trees of \( G \)
end.

Figure 1 illustrates four independent spanning trees found by the algorithm for the 4-canonical decomposition depicted in Fig. 3 (c).
4. Correctness of the Algorithm

In this section we verify the correctness of our algorithm. Throughout this section, let \( \Pi = (W_1, W_2, \cdots, W_m) \) be a 4-canonical decomposition of \( G \). We first prove the following lemma.

**Lemma 3** Let \( 2 \leq i \leq m - 1 \), and let \( T_{r_l}^i \) be a subgraph of \( G_i \) induced by the right legs of all vertices in \( G_i \) except \( r_b \). Then \( T_{r_l}^i \) is a spanning tree of \( G_i \).

**Proof.** We prove, by induction on \( i \), the proposition that \( T_{r_l}^i \) is a spanning tree of \( G_i \).

The definition of a 4-canonical decomposition implies that \( G_2 \) is biconnected and all vertices in \( W_2 \) consecutively appear on \( C_o(G_2) \). Therefore, \( W_2 \) satisfies condition (b), \( G_2 \) is a simple cycle on the contour of the inner face containing \( r_b \) in \( G'' \), and \( W_2 \) consists of all vertices \( u_1, u_2, \cdots, u_h \) consecutively appearing clockwise on \( C_o(G_2) \) between \( r_2 = u_1 \) and \( r_1 = u_h \). (See Fig. 3 (c).) Since \( rl(u_1) = (u_1, u_2), \) \( rl(u_2) = (u_2, u_3), \cdots, rl(u_h) = (u_h, r_b) \), \( T_{r_l}^1 \) is a path \( u_1, u_2, \cdots, u_h, r_b \) and hence \( T_{r_l}^1 \) is a spanning tree of \( G_2 \). Therefore the proposition holds for \( i = 2 \).

Assume inductively that the proposition holds for \( i, 2 \leq i \leq m - 2 \), and we shall prove that the proposition holds for \( i + 1 \). There are the following three cases to consider.

**Case 1:** \( W_{i+1} \) satisfies condition (a). (See Fig. 5.)

In this case, \( W_{i+1} = \{ u \} \) for a vertex \( u \in V'' \). \( T_{r_l}^{i+1} \) is obtained from the spanning tree \( T_{r_l}^i \) of \( G_i \) by adding the right leg \( rl(u) \) to \( T_{r_l}^i \), and the right foot \( rf(u) \) of \( u \) is a vertex in \( T_{r_l}^i \). Therefore \( T_{r_l}^{i+1} \) is a spanning tree of \( G_{i+1} \), as illustrated in Fig. 8 (a) where \( T_{r_l}^{i+1} \) is drawn by thick lines.

**Case 2:** \( W_{i+1} \) satisfies condition (b). (See Fig. 6.)

Let \( W_{i+1} = \{ u_1, u_2, \cdots, u_h \} \), where \( u_1, u_2, \cdots, u_h \) consecutively appear clockwise on \( C_o(G_{i+1}) \) in this order. Let \( u_{h+1} = rf(u_h) \), then \( u_{h+1} \) is a vertex in \( T_{r_l}^i \). \( T_{r_l}^{i+1} \) is obtained from \( T_{r_l}^i \) by adding the \( h \) edges \( (u_1, u_2), (u_2, u_3), \cdots, (u_h, u_{h+1}) \) to \( T_{r_l}^i \). Thus \( T_{r_l}^{i+1} \) is a spanning tree of \( G_{i+1} \) as illustrated in Fig. 8 (b).

**Case 3:** \( W_{i+1} \) satisfies condition (c). (See Fig. 7.)

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Fig. 8. Illustration for Lemma 3.
Let \( W_{i+1} = \{u_1, u_2, \ldots, u_h\} \), where \( u_1, u_2, \ldots, u_h \) consecutively appear clockwise on \( C_0(G_{i+1}) \) in this order. Then \( T^{i+1}_{rl} \) is obtained from \( T^i_{rl} \) by adding the right legs \( rl(u_1), rl(u_2), \ldots, rl(u_h) \) to \( T^i_{rl} \), and all the right feet \( rf(u_1), rf(u_2), \ldots, rf(u_h) \) are vertices in \( T^i_{rl} \). Therefore \( T^{i+1}_{rl} \) is a spanning tree of \( G_{i+1} \) as illustrated in Fig. 8(c).

We then have the following lemma.

**Lemma 4** \( T_{rl}, T_{ll}, T_{la} \) and \( T_{ra} \) are spanning trees of \( G \) if both \( r_b \) and \( r_t \) are regarded as \( r \).

**Proof.** By Lemma 3 \( T^{m-1}_{rl} \) is a spanning tree of \( G_{m-1} \). Clearly \( T_{rl} = T^{m-1}_{rl} \). Hence \( T_{ll} \) is a spanning tree of \( G \) if \( r_b \) in \( G_{m-1} \) is regarded as \( r \) in \( G \).

Similarly, \( T_{ll}, T_{la} \) and \( T_{ra} \) are spanning trees of \( G \).

Thus it suffices to prove that the four spanning trees \( T_{rl}, T_{ll}, T_{la} \) and \( T_{ra} \) are independent. We first prove that any pair of an arm path and a leg path are internally disjoint. More precisely, we have the following lemma.

**Lemma 5** For any vertex \( v \in V'' - \{r_b, r_t\} \), each of the four pairs \((P_{rl}(v), P_{la}(v)), (P_{rl}(v), P_{ra}(v)), (P_{ll}(v), P_{la}(v)), \) and \((P_{ll}(v), P_{ra}(v))\) of paths are internally disjoint if both \( r_b \) and \( r_t \) are regarded as \( r \).

**Proof.** We prove only that paths \( P_{rl}(v) \) and \( P_{la}(v) \) are internally disjoint in \( G'' \) if both \( r_b \) and \( r_t \) are regarded as \( r \). Let \( P_{rl}(v) = v, u_1, u_2, \ldots, u_j, \) and \( P_{la}(v) = v, u_1, u_2, \ldots, u_j', \) where \( u_j = r_b \) and \( u_j' = r_t \). By Lemma 2 both \( P_{rl}(v) \) and \( P_{la}(v) \) are simple paths. By (S1) and (S2), we have \( \text{rank}(u_j) \leq \text{rank}(u_{j-1}) \leq \cdots \leq \text{rank}(u_1) \leq \text{rank}(v) \leq \text{rank}(w_i) \leq \cdots \leq \text{rank}(w_j') \). We furthermore have \( \text{rank}(u_1) < \text{rank}(u_1) \) by (S4). Therefore \( P_{rl}(v) \) and \( P_{la}(v) \) have no common intermediate vertex. Hence they are internally disjoint if both \( r_b \) and \( r_t \) are regarded as \( r \).

We then prove that both the pair of arm paths and the pair of leg paths are internally disjoint. We now have the following lemma.

**Lemma 6** Let \( u \) be any vertex in \( G'' \) other than \( r_b \) and \( r_t \). Let \( N(u) = \{v_1, v_2, \ldots, v_{d(u)}\} \), where \( v_1 = \text{l}(u) \). Let \( v_s = \text{r}(u) \), where \( 4 \leq s \leq d(u) \). (See Fig. 9.) If \( \text{r}(v_j) = \text{l}(v_{j'}) = u \) for two distinct indices \( j \) and \( j' \), \( 1 \leq j, j' \leq s \), then \( j < j' \).

**Proof.** Assume, on the contrary, that \( \text{r}(v_j) = \text{l}(v_{j'}) = u \), \( 1 \leq j, j' \leq s \), but \( j > j' \). Since neither \( r_b \) nor \( r_t \) has any leg, we have \( v_j, v_{j'} \neq r_b, r_t \). Let \( k = \text{rank}(v_j) \) and \( k' = \text{rank}(v_{j'}) \), then

\[
\text{rank}(v_1), \text{rank}(v_s) \leq \text{rank}(u) \leq k, k' \]

and

\[
2 \leq k, k' \leq m - 1.
\]

There are the following three cases to consider.
Case 1: \( k = k' \).

In this case, \( v_j, v_{j'} \in W_k \) and hence \( |W_k| \geq 2 \). Therefore \( W_k \) satisfies either condition (b) or (c). Consider first the case where \( W_k \) satisfies condition (b). (See Fig. 6.) Then, since \( j > j' \) and \( rf(v_j) = lf(v_{j'}) = u \), one can easily know that \( rank(u) < k, u \notin W_k, v_{j'} \) is the first vertex in \( W_k \), and \( v_j \) is the last vertex in \( W_k \). Thus \( u \) is a cut vertex of \( G_k \) and hence \( G_k \) is not biconnected, contradicting the definition of the 4-canonical decomposition. Consider next the case where \( W_k \) satisfies condition (c). Then \( u, v_{j'}, v_{j'+1}, \ldots, v_j, u \) is a simple cycle in \( G \); the proper outside of the cycle contains root \( r \); and the proper inside contains at least one vertex of \( G \), say \( lf(v_j) \neq r \). Thus the removal of three vertices \( u, v_{j'} \) and \( v_j \) disconnects \( G \), contradicting the 4-connectivity of \( G \).

Case 2: \( k < k' \).

Since \( rank(v_j) = k, v_j \) is on \( C_o(G_k) \). However, \( rl(v_j) = (v_j, u) \) is not always on \( C_o(G_k) \). Consider first the case where \( (v_j, u) \) is not on \( C_o(G_k) \). Then \( W_k \) must satisfy condition (c) and \( v_j \) must not be the last vertex in \( W_k \); otherwise, \( rl(v_j) \) would be on \( C_o(G_k) \). Since \( (v_j, u) \) is not on \( C_o(G_k) \), \( u \) is in the proper inside of \( C_o(G_k) \). Since \( k < k' \), \( G_{k'} \) is obtained from \( G_k \) by adding the vertices in \( W_{k+1} \cup W_{k+2} \cup \cdots \cup W_{k'} \) to \( G_k \). All these added vertices are outside \( C_o(G_k) \) in the plane graph \( G_{k'} \). In particular, the vertex \( v_{j'} \in W_{k'} \) is not on \( C_o(G_k) \) but on \( C_o(G_{k'}) \). Therefore \( v_{j'} \) cannot have \( u \) in the proper inside of \( C_o(G_k) \) as a foot, contradicting \( lf(v_{j'}) = u \). Consider next the case where \( rl(v_j) = (v_j, u) \) is on \( C_o(G_k) \). All neighbors of \( v_j \) clockwise appearing around \( v_j \) between \( rf(v_j) = u \) and \( lf(v_j) \) have ranks \( \leq k \) and are contained in \( G_k \). Therefore \( v_j \) precedes \( u \) on \( C_o(G_k) \). Since \( 2 \leq k, G_k \) is biconnected and hence \( C_o(G_k) \) is a simple cycle. Since \( rank(v_1), rank(v_s) \leq k, v_1 \) and \( v_s \) are contained in \( G_k \). Let \( v_t \) be the vertex succeeding \( u \) on \( C_o(G_k) \), then \( j + 1 \leq t \leq s \), and \( v_j, u, v_t \) consecutively appear
clockwise on the simple cycle $C_o(G_k)$ in this order. Therefore, the edge $(v_j', u)$ added to $G_k$ must appear around $u$ between $(v_j, u)$ and $(v_t, u)$ in $G_{k'}$ and hence in $G''$. Thus $v_j, v_j', v_t$ must appear in $N(u)$ in this order. Therefore $j < j'$, contradicting the assumption that $j > j'$.

**Case 3:** $k > k'$.

Consider first the case where $ll(v_j') = (v_j', u)$ is not on $C_o(G_{k'})$. Then $W_{k'}$ must satisfy condition (c) and $v_j'$ must not be the first vertex in $W_{k'}$. Since $(v_j', u)$ is not on $C_o(G_{k'})$, $u$ is in the proper inside of $C_o(G_{k'})$. Since $k > k'$, $G_k$ is obtained from $G_{k'}$ by adding the vertices in $W_{k' + 1} \cup W_{k' + 2} \cup \cdots \cup W_{k}$ to $G_{k'}$. Therefore $v_j \in W_k$ is not on $C_o(G_{k'})$ but on $C_o(G_k)$, and hence $v_j$ cannot have $u$ as a foot, contradicting $rf(v_j) = u$. Consider next the case where $ll(v_j') = (v_j', u)$ is on $C_o(G_{k'})$. Then $v_j'$ succeeds $u$ on $C_o(G_{k'})$. Let $v_t$ be the vertex preceding $u$ on $C_o(G_{k'})$, then $1 \leq t \leq j' - 1$, and $v_t, v_j, v_j'$ appear in $N(u)$ in this order. Therefore $j < j'$, contradicting the assumption that $j > j'$.

We then have the following lemma.

**Lemma 7** Let $v$ be any vertex in $G''$ other than $r_b$ and $r_t$. Then both the pair of leg paths $P_{rl}(v)$ and $P_{ll}(v)$ and the pair of arm paths $P_{la}(v)$ and $P_{ra}(v)$ are internally disjoint.

![Illustration for the proof of Lemma 7](image_url)
Proof. We prove only that $P_{rl}(v)$ and $P_{ul}(v)$ are internally disjoint. Suppose for a contradiction that $P_{rl}(v)$ and $P_{ul}(v)$ share an intermediate vertex. Let $P_{rl}(v) = v_1, v_2, \ldots, v_j$ and $P_{ul}(v) = w_1, w_2, \ldots, w_j'$, where $v = v_1 = w_1$ and $v_j = w_j'$. In Fig. 10 $P_{rl}(v)$ is drawn by thick line and $P_{ul}(v)$ by thin lines. Let $u$ be the first intermediate vertex shared by the two paths $P_{rl}(v)$ and $P_{ul}(v)$ starting from $v$. Then $u = v_i = w_k$ for some $i$, $2 \leq i \leq j - 1$, and $k$, $2 \leq k \leq j' - 1$, and vertices $v_2, v_3, \ldots, v_{i-1}, w_2, w_3, \ldots, w_{k-1}$ are distinct with each other. Furthermore, either $w_{k-1}, v_{i-1}, w_{k+1}, v_{i+1}$ appear clockwise around $u$ in this order as illustrated in Fig. 10 (a), or $w_{k-1}, v_{i-1}, v_{i+1}, w_{k+1}$ appear clockwise around $u$ in this order as illustrated in Fig. 10 (b). If $w_{k-1}, v_{i-1}, w_{k+1}, v_{i+1}$ appear clockwise around $u$ in this order, then $w_{k+1} = lf(u), v_{i+1} = rf(u), lh(u), rh(u)$ must appear clockwise around $u$ in this order. This is a contradiction, because by the property (S3) $lf(u), lh(u), rh(u)$ and $rf(u)$ appear clockwise around $u$ in this order. Thus one may assume that $w_{k-1}, v_{i-1}, v_{i+1}, w_{k+1}$ appear clockwise around $u$ in this order. Then $w_{k+1} = lf(u), w_{k-1}, v_{i-1}, v_{i+1} = rf(u)$ appear clockwise around $u$ in this order. However, $rf(v_{i-1}) = lf(w_{k-1}) = u$, contradicting Lemma 6. □

By Lemmas 4, 5 and 7 we have the following lemma.

Lemma 8 $T_{rl}, T_{ul}, T_{la}$ and $T_{ra}$ are four independent spanning trees of $G$ rooted at $r$.

A plane embedding of $G$ can be constructed in linear time [12]. A 4-canonical decomposition $\Pi$ can be found in linear time [13]. $T_{rl}, T_{ul}, T_{la}$ and $T_{ra}$ can be found in linear time. Thus the running time of Algorithm FourTrees is $O(n)$. Thus we have the following theorem.

Theorem 1 Four independent spanning trees rooted at any designated vertex can be found in linear time for any 4-connected plane graph.

5. Conclusion

In this paper we give a linear-time algorithm to find four independent spanning trees of a 4-connected planar graph rooted at any designated vertex. Our algorithm is faster than the best known $O(n^3)$ algorithm in [7], and the complexity is optimal within a constant factor. Using four independent spanning trees, one can efficiently solve the 4-paths query problem for 4-connected planar graphs, which asks to find four internally disjoint paths connecting a specified pair of vertices. It is remained as a future work to find a linear-time algorithm for a larger class of graphs, say 4-connected graphs which are not always planar.

References


