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A LINEAR-TIME ALGORITHM TO FIND FOUR INDEPENDENT SPANNING TREES IN FOUR CONNECTED PLANAR GRAPHS

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ABSTRACT

Given a graph G, a designated vertex r and a natural number k, we wish to find k "independent" spanning trees of G rooted at r, that is, k spanning trees such that the k paths connecting r and any vertex v in the k trees are internally disjoint. In this paper we give a linear-time algorithm to find four independent spanning trees in a 4-connected planar graph.

Keywords: Graph, independent spanning tree, algorithm.

1. Introduction

Given a graph $G = (V, E)$, a designated vertex $r \in V$ and a natural number k, we wish to find k spanning trees $T_1, T_2, \cdots, T_k$ of G such that, for any vertex v, the k paths connecting r and v in $T_1, T_2, \cdots, T_k$ are internally disjoint, that is, any two of them have no common intermediate vertices. Such k trees are called *independent spanning trees of G (rooted at r)*. Four independent spanning trees are drawn by thick lines in Fig. 1. Note that independent spanning trees are not always edge-disjoint. Independent spanning trees have applications to fault-tolerant protocols in networks [1, 5, 8, 14].

It is conjectured that, for any $k \geq 1$, every k-connected graph G has k independent spanning trees rooted at any vertex [10, 15]. If G is a biconnected graph, then one can find two independent spanning trees in linear time by means of the st-numbering [2, 8]. If G is triconnected graph, then one can find three independent

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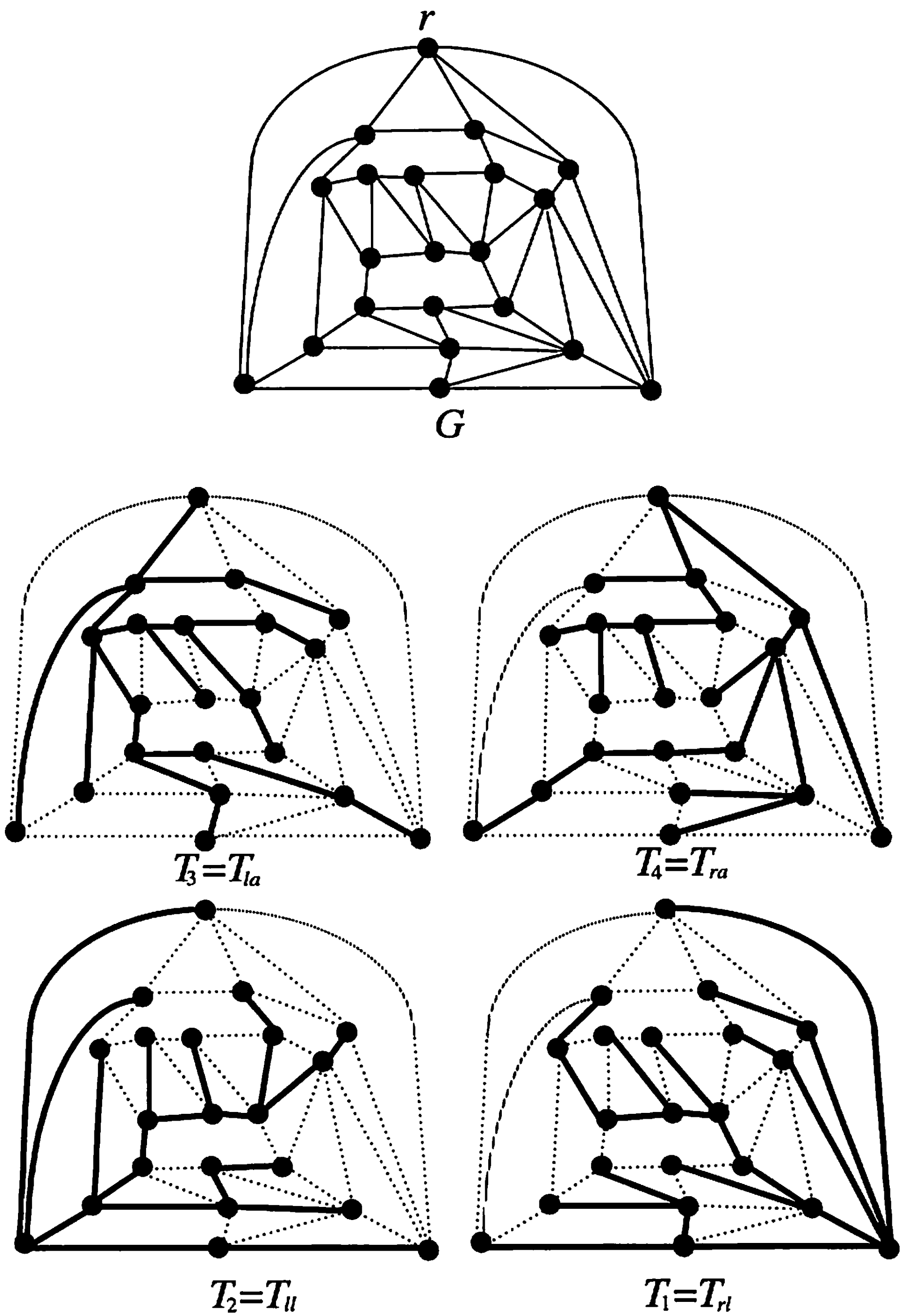

Fig. 1. Four independent spanning trees T_1, T_2, T_3 and T_4 of a graph G rooted at r.

spanning trees in $O(n^2)$ time by means of the ear-decomposition, where n is the number of vertices of G [2, 4]. If G is a triconnected planar graph, then one can find three independent spanning trees in linear time by means of the canonical ordering [2]. On the other hand, it is not known whether the conjecture holds for $k \geq 4$ in general. However, Huck has recently shown that the conjecture holds for $k = 4$ if G is planar, proving that every 4-connected planar graph has four independent spanning trees [7]. The proof in [7] yields an algorithm to actually find four independent spanning trees, but it takes time $O(n^3)$.

In this paper we give a simple linear-time algorithm to find four independent spanning trees of a 4-connected planar graph. Our algorithm is based on the "4-canonical decomposition" of a 4-connected planar graph [13], which is a generalization of the *st*-numbering [6], the canonical ordering [3] and the canonical 4-ordering [9].

The remainder of the paper is organized as follows. In Section 2 we introduce some definitions. In Section 3 we present our algorithm to find four independent spanning trees. In Section 4 we verify the correctness of our algorithm. Finally we conclude in Section 5 with some general comments and an open problem. An early version of the paper was presented in [11].

2. Preliminaries

In this section we introduce some definitions.

Throughout the paper, we consider a simple connected graph G. A graph G with vertex set V and edge set E is denoted by $G = (V, E)$. We denote by n the number of vertices in G, and assume that $n \geq 4$. An edge joining vertices u and v is denoted by (u, v). The *degree* of a vertex v in G is the number of neighbors of v in G and is denoted by $d(v, G)$ or simply by $d(v)$. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. A graph G is *k-connected* if $\kappa(G) \geq k$. A *path* passing through vertices $v_1, v_2, \cdots, v_l$ in this order is denoted by the sequence $v_1, v_2, \cdots, v_l$, where (v_i, v_{i+1}) is an edge for every i, $1 \leq i \leq l-1$. We say that a path is *simple* if all the vertices are distinct from each other. We also say that a set of paths having common start and end vertices are *internally disjoint* if any pair of paths in the set have no common intermediate vertex. A *cycle* is a path beginning and ending with the same vertex. A cycle $v_1, v_2, \cdots, v_l (= v_1)$ is *simple* if all vertices $v_1, v_2, \cdots, v_{l-1}$ and all edges are distinct with each other.

A graph is *planar* if it can be embedded on the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane graph* is a planar graph with a fixed plane embedding. The *contour* $C_o(G)$ of a plane connected graph G is the cycle on the outer face. We always regard $C_o(G)$ as a *clockwise* cycle. Thus we write $C_o(G) = w_1, w_2, \cdots, w_l$ if the vertices $w_1, w_2, \cdots, w_l$ on $C_o(G)$ appear clockwise in this order, where $w_l = w_1$. If G is biconnected, then $C_o(G)$ is a simple cycle.

3. Algorithm

In this section we give our algorithm to find four independent spanning trees of a 4-connected planar graph G rooted at any designated vertex r.

Our main idea is very simple as follows. For each vertex v, we assign four edges incident to v as the right leg $rl(v)$, left leg $ll(v)$, left arm $la(v)$, and right arm $ra(v)$ of v; they appear clockwise around v in this order as illustrated in Fig. 2. Intuitively, the right leg $rl(v)$ is the edge joining v and the "lower rightmost neighbor" of v in a plane embedding of G having the root r at the top. Similarly, the left leg $ll(v)$ is the edge joining v and the "lower leftmost neighbor," the left arm $la(v)$ is the edge joining v and the "upper leftmost neighbor," the right arm $ra(v)$ is the edge joining v and the "upper rightmost neighbor." The four subgraphs of G, each of which is induced by all right legs, all left legs, all left arms, or all right arms, are four independent spanning trees of G, as seen later. The end of right leg $rl(v)$ other than v is called the right foot $rf(v)$ of v. Similarly the left foot $lf(v)$, the left hand $lh(v)$ and the right hand $rh(v)$ are defined. The four neighbors $rf(v), lf(v), lh(v)$ and $rh(v)$ of v appear clockwise around v in this order.

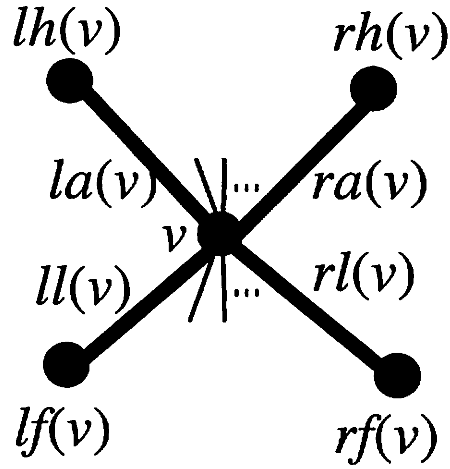

Fig. 2. Legs, arms, feet, and hands of a vertex v.

The assignment above can be done by means of the "4-canonical decomposition" of a graph, which is defined as follows. Given a 4-connected planar graph $G = (V, E)$ and a vertex $r \in V$ designated as a root, we first find a plane embedding of G in which r is located on the top of $C_o(G)$ [12]. Let $r_1, r_2, \cdots, r_{d(r)}$ be the neighbors of r in G. Since G is 4-connected, $d(r) \geq 4$. Let G' be the plane graph obtained from G by deleting root r and all the edges $(r, r_1), (r, r_2), \cdots, (r, r_{d(r)})$ incident to r. Fig. 3 (a) depicts G, and Fig. 3 (b) depicts G'. One may assume that $r_1, r_2, \cdots, r_{d(r)}$ appear clockwise on $C_o(G')$ in this order. We add to G' two new

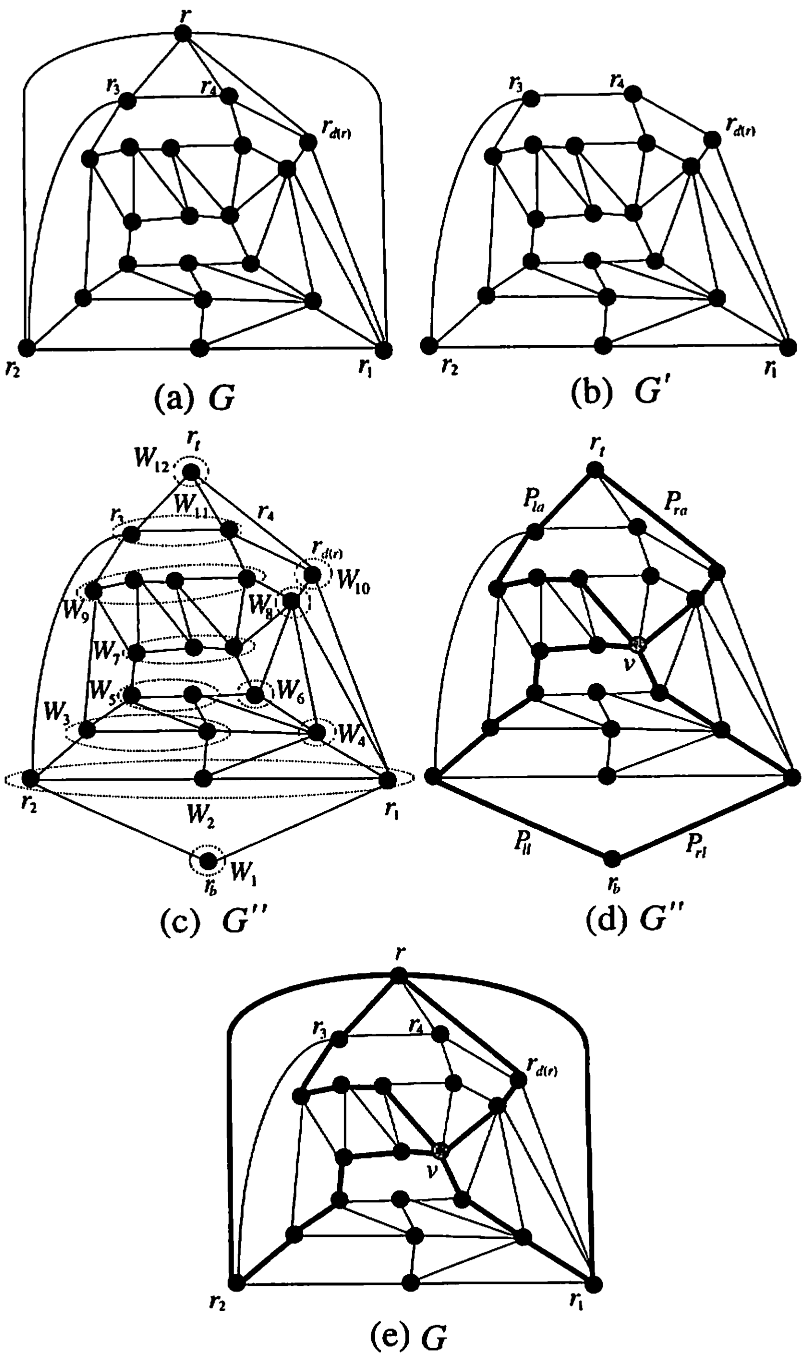

Fig. 3. (a) Four-connected plane graph G, (b) plane graph G', (c) plane graph G'', (d) four leg and arm paths starting from a vertex v in G'', and (e) four internally disjoint paths between r and v in G.

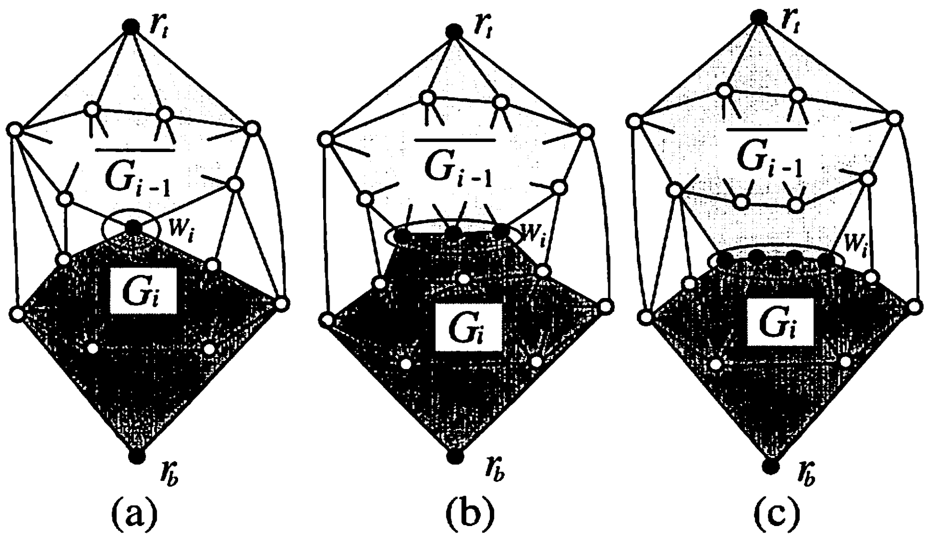

Fig. 4. Three conditions (a), (b) and (c).

vertices, the *bottom root* r_b and the *top root* r_t. We then join r_b with r_1 and r_2, and join r_t with $r_3, r_4, \cdots, r_{d(r)}$, as illustrated in Fig 3 (c). Let $G'' = (V'', E'')$ be the resulting plane graph. Then $V'' = V \bigcup \{r_b, r_t\} - \{r\}$, and vertices r_1, r_b, r_2, r_3, r_t and $r_{d(r)}$ appear on $C_o(G'')$ clockwise in this order. Let m be a natural number, and let $\Pi = (W_1, W_2, \cdots, W_m)$ be a partition of set V'' to m subsets $W_1, W_2, \cdots, W_m$ of V'', where $W_1 = \{r_b\}$ and $W_m = \{r_t\}$. In Fig. 3 (c) each set W_i is indicated by an oval drawn in a dotted line. We say that the *rank* of a vertex $v \in V''$, denoted by $rank(v)$, is i if $v \in W_i$ for i, $1 \leq i \leq m$. We denote by G_i, $1 \leq i \leq m$, the plane subgraph of G'' induced by all vertices of rank $\leq i$. We denote by $\overline{G_{i-1}}$, $1 \leq i \leq m$, the plane subgraph of G'' induced by all vertices of rank $\geq i$. In Fig. 4, G_i is darkly shaded, while $\overline{G_{i-1}}$ is lightly shaded. We assume that both G_i and $\overline{G_{i-1}}$ are biconnected for each i, $2 \leq i \leq m-1$. We furthermore assume that if $W_i = \{u_1, u_2, \cdots, u_h\}$ then all vertices $u_1, u_2, \cdots, u_h$ in W_i consecutively appear clockwise on $C_o(G_i)$ in this order for every i, $1 \leq i \leq m$. Thus, for each i, $2 \leq i \leq m$, all vertices in W_i are located outside $C_o(G_{i-1})$ in the plane graph G_i. Then the partition Π is called a *4-canonical decomposition of* G (*rooted at* r) if, for each i, $2 \leq i \leq m-1$, one of the following three conditions holds:

(a) W_i consists of exactly one vertex having at least two neighbors of rank $> i$ and at least two neighbors of rank $< i$. (See Fig. 4 (a).)

(b) W_i consists of two or more vertices; every vertex in W_i has at least two neighbors of rank $> i$; each of the first and the last vertices in W_i has exactly one neighbor of rank i and exactly one neighbor of rank $< i$; and each of the other vertices in W_i has exactly two neighbors of rank i, and has no neighbor of rank $< i$. (See Fig. 4 (b).)

(c) W_i consists of two or more vertices; every vertex in W_i has at least two neighbors of rank $< i$; each of the first and the last vertices in W_i has exactly one neighbor of rank i and exactly one neighbor of rank $> i$; and each of the other vertices in W_i has exactly two neighbors of rank i, and has no neighbor of rank $> i$. (See Fig. 4 (c).)

Note that conditions (b) and (c) are symmetric. Fig. 3 (c) illustrates a 4-canonical decomposition $\Pi = (W_1, W_2, \cdots, W_{12})$ of the graph G in Fig. 3 (a). Subsets W_4, W_6, W_8 and W_{10} satisfy condition (a), W_2, W_3 and W_7 satisfy condition (b), and W_5, W_9 and W_{11} satisfy condition (c).

The vertex of rank 1, i.e. the bottom root r_b, has exactly two neighbors of larger ranks in G'', while the vertex of rank m, i.e. the top root r_t, has at least two neighbors of smaller ranks. Each vertex v of rank i, $2 \le i \le m-1$, has at least two neighbors of ranks $\ge i$ and has at least two neighbors of ranks $\le i$ whichever condition (a), (b) or (c) holds for i; and furthermore each vertex v of rank i has at most two neighbors of rank i, and if v has exactly two neighbors of rank i then the ranks of all the other neighbors of v are either greater than i or smaller than i.

The 4-canonical decomposition is a generalization of the "*st*-numbering" [6], the "canonical decomposition" [3] and the "canonical 4-ordering" [9]. Although the definition of a 4-canonical decomposition above is slightly different from that in [13], they are effectively equivalent with each other and we have the following lemma.

Lemma 1 Let $G = (V, E)$ be a 4-connected plane graph, and let r be a designated vertex on $C_o(G)$. Then G has a 4-canonical decomposition Π rooted at r. Furthermore Π can be found in linear time.

Proof. Similar to the proof of Lemma 3 in [13]. □

Let v be any vertex in G'' other than r_b and r_t. We write $N(v) = \{v_1, v_2, \cdots, v_{d(v,G'')}\}$ if the neighbors $v_1, v_2, \cdots, v_{d(v,G'')}$ of v in G'' appear clockwise around v in this order. All v's neighbors having ranks $\ge rank(v)$ consecutively appear around v, and so do all neighbors having ranks $\le rank(v)$. This is the reason why one can define legs and arms of v as follows. We assign four distinct edges incident to v in G'' as *the right leg* $rl(v)$, *the left leg* $ll(v)$, *the left arm* $la(v)$ and *the right arm* $ra(v)$ of v in a way that the following four statements (S1)–(S4) hold. (See Fig. 2.)

(S1) The two feet of v have ranks $\le rank(v)$.

(S2) The two hands of v have ranks $\ge rank(v)$.

(S3) The right arm, the right leg, the left leg and the left arm of v appear clockwise around v in this order; in particular, the right arm precedes the right leg, and the left leg precedes the left arm.

(S4) For any foot u and any hand w of v, $rank(u) < rank(w)$.

For each vertex $v \in W_i$, $2 \le i \le m-1$, the definition above assigns four edges as legs and arms of v, as follows.

Case 1: W_i satisfies condition (a). (See Fig. 5.)

In this case $W_i = \{u\}$ for a vertex $u \in V''$. Let v_1 be the vertex preceding u on $C_o(G_i)$, and let $N(u) = \{v_1, v_2, \cdots, v_{d(u)}\}$. Then $ll(u) = (u, v_1)$ and $la(u) = (u, v_2)$. Let v_s, $1 \le s \le d(u)$, be the vertex succeeding u on $C_o(G_i)$, then $ra(u) = (u, v_{s-1})$ and $rl(u) = (u, v_s)$.

Case 2: W_i satisfies condition (b). (See Fig. 6.)

Let $W_i = \{u_1, u_2, \cdots, u_h\}$, where $u_1, u_2, \cdots, u_h$ consecutively appear on $C_o(G_i)$ clockwise in this order. Let u_0 be the vertex preceding u_1 on $C_o(G_i)$, and let u_{h+1} be the vertex succeeding u_h on $C_o(G_i)$. Then, for each vertex $u_j \in W_i$, $ll(u_j) = (u_j, u_{j-1})$ and $rl(u_j) = (u_j, u_{j+1})$. Let $N(u_j) = \{v_1, v_2, \cdots, v_{d(u_j)}\}$, $v_1 = u_{j-1}$ and $v_{d(u_j)} = u_{j+1}$, then $la(u_j) = (u_j, v_2)$ and $ra(u_j) = (u_j, v_{d(u_j)-1})$.

Case 3: W_i satisfies condition (c). (See Fig. 7.)

Let $W_i = \{u_1, u_2, \cdots, u_h\}$, where $u_1, u_2, \cdots, u_h$ consecutively appear on $C_o(G_i)$ clockwise in this order. Let u_0 be the vertex succeeding u_1 on $\overline{C_o(G_{i-1})}$, and let u_{h+1} be the vertex preceding u_h on $\overline{C_o(G_{i-1})}$. Then, for each vertex $u_j \in W_i$, $ra(u_j) = (u_j, u_{j+1})$ and $la(u_j) = (u_j, u_{j-1})$. Let $N(u_j) = \{v_1, v_2, \cdots, v_{d(u_j)}\}$, $v_1 = u_{j+1}$, and $v_{d(u_j)} = u_{j-1}$, then $rl(u_j) = (u_j, v_2)$ and $ll(u_j) = (u_j, v_{d(u_j)-1})$.

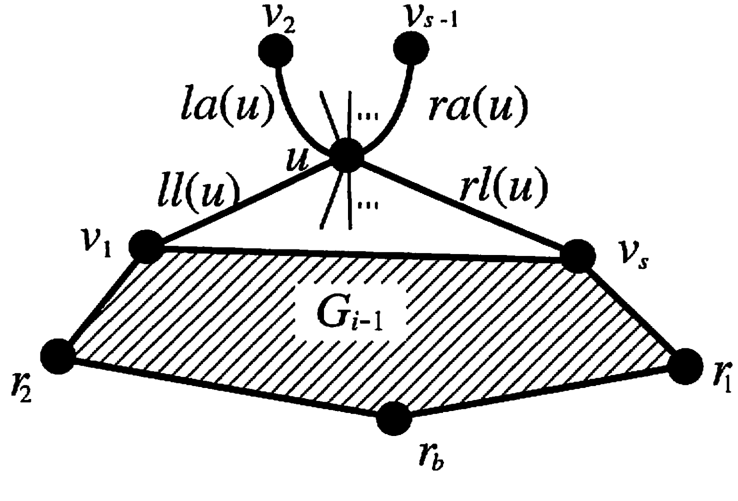

Fig. 5. Assignment for Case 1.

For each vertex $v \in V'' - \{r_b, r_t\}$, the *right leg path* $P_{rl}(v) = v_1, v_2, \cdots, v_j$ is defined to be the path connecting $v = v_1$ and $r_b = v_j$ such that $rl(v_i) = (v_i, v_{i+1})$ for every i, $1 \le i \le j-1$. Similarly, for each v, we define the *left leg path* $P_{ll}(v)$ connecting v and r_b, the *left arm path* $P_{la}(v)$ connecting v and r_t, the *right arm path* $P_{ra}(v)$ connecting v and r_t. Intuitively, the right leg path goes "southeast," the left

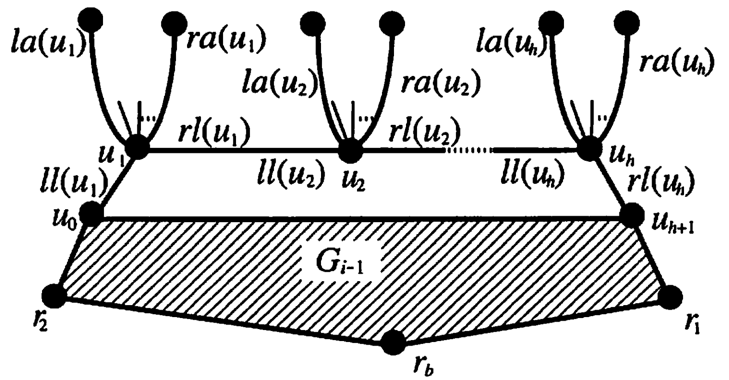

Fig. 6. Assignment for Case 2.

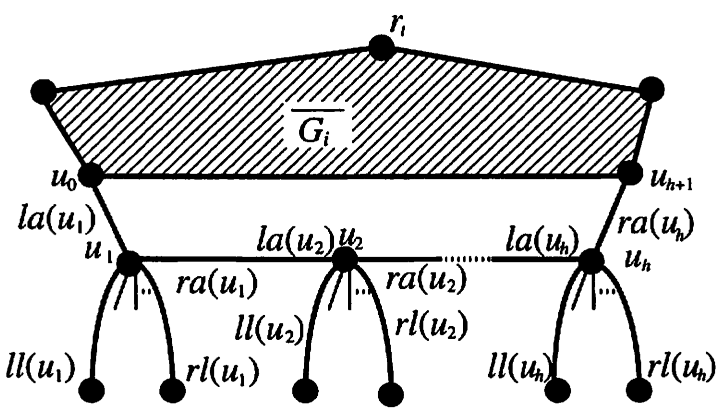

Fig. 7. Assignment for Case 3.

leg path goes "southwest," the left arm path goes "northwest," and the right arm path goes "northeast" if r_t and r_b are located at the top and bottom, respectively, in the plane graph G''. In Fig. 3 (d) the four paths starting from a vertex v in G'' are drawn by thick lines, and in Fig. 3 (e) the corresponding four internally disjoint paths between r and v in G are drawn by thick lines.

We have the following lemma.

Lemma 2 $P_{rl}(v)$ and $P_{ll}(v)$ are simple paths connecting v and r_b in G'', and $P_{la}(v)$ and $P_{ra}(v)$ are simple paths connecting v and r_t in G''.

Proof. We prove only that $P_{rl}(v)$ is a simple path connecting v and r_b in G''. Proofs for the other paths are similar. Let $P_{rl}(v) = v_1, v_2, \cdots, v_j$ where $v_1 = v$. The property (S1) of legs implies that $rank(v_1), rank(v_2), \cdots, rank(v_j)$ are nonincreasing, that is

$$rank(v_1) \geq rank(v_2) \geq \cdots \geq rank(v_j).$$

We first show that $P_{rl}(v)$ is a simple path, that is, any two vertices v_i and $v_{i'}$ in $P_{rl}(v)$ are distinct. One may assume that $1 \leq i < i' \leq j$. Let $k = rank(v_i)$ and $k' = rank(v_{i'})$, then $k \geq k'$. If $k > k'$, then obviously v_i and $v_{i'}$ are distinct. Thus one may assume that $k = k'$. Then $v_i, v_{i'} \in W_k$, and hence W_k satisfies condition (b); otherwise, W_k would satisfy condition (a) or (c), and hence $|W_k| = 1$. (See Fig. 7.) Since W_k satisfies condition (b), the subpath of $P_{rl}(v)$ between v_i and $v_{i'}$ is a simple path all vertices of which have rank k. (See Fig. 6.) Therefore v_i and $v_{i'}$ are distinct.

We next show that $P_{rl}(v)$ ends at r_b. Every vertex in G'' except r_b and r_t has a right leg. Vertex r_b is a right foot of r_1, but r_t is not a right foot of any vertex in G''. Furthermore $P_{rl}(v)$ is a finite simple path. Therefore $P_{rl}(v)$ must end at r_b. □

As seen later in Lemmas 5 and 7, these four paths starting from v have no common intermediate vertex, and hence these four paths are internally disjoint if the bottom root r_b and the top root r_t in G'' are regarded as the root r in G.

We are now ready to give our algorithm.

Procedure FourTrees(G, r)
begin
Find a plane embedding of G such that $r \in C_o(G)$;
Find a 4-canonical decomposition $\Pi = (W_1, W_2, \cdots, W_m)$ of G rooted at r;
Find $rl(v), ll(v), la(v)$ and $ra(v)$ for each vertex $v \in V'' - \{r_b, r_t\}$;
Let T_{rl}, T_{ll}, T_{la} and T_{ra} be subgraphs of G'' induced by the right legs, left legs, left arms and right arms of all vertices in $V'' - \{r_b, r_t\}$, respectively;
Regard vertex r_b in trees T_{rl} and T_{ll} as vertex r;
Regard vertex r_t in trees T_{la} and T_{ra} as vertex r;
return T_{rl}, T_{ll}, T_{la} and T_{ra} as four independent spanning trees of G
end.

Figure 1 illustrates four independent spanning trees found by the algorithm for the 4-canonical decomposition depicted in Fig. 3 (c).

4. Correctness of the Algorithm

In this section we verify the correctness of our algorithm. Throughout this section, let $\Pi = (W_1, W_2, \cdots, W_m)$ be a 4-canonical decomposition of G. We first prove the following lemma.

Lemma 3 Let $2 \le i \le m-1$, and let T^i_{rl} be a subgraph of G_i induced by the right legs of all vertices in G_i except r_b. Then T^i_{rl} is a spanning tree of G_i.

Proof. We prove, by induction on i, the proposition that T^i_{rl} is a spanning tree of G_i.

The definition of a 4-canonical decomposition implies that G_2 is biconnected and all vertices in W_2 consecutively appear on $C_o(G_2)$. Therefore, W_2 satisfies condition (b), G_2 is a simple cycle on the contour of the inner face containing r_b in G'', and W_2 consists of all vertices $u_1, u_2, \cdots, u_h$ consecutively appearing clockwise on $C_o(G_2)$ between $r_2 = u_1$ and $r_1 = u_h$. (See Fig. 3 (c).) Since $rl(u_1) = (u_1, u_2), rl(u_2) = (u_2, u_3), \cdots, rl(u_h) = (u_h, r_b)$, T^2_{rl} is a path $u_1, u_2, \cdots, u_h, r_b$ and hence T^2_{rl} is a spanning tree of G_2. Therefore the proposition holds for $i = 2$.

Assume inductively that the proposition holds for i, $2 \le i \le m-2$, and we shall prove that the proposition holds for $i+1$. There are the following three cases to consider.

Case 1: W_{i+1} satisfies condition (a). (See Fig. 5.)

In this case, $W_{i+1} = \{u\}$ for a vertex $u \in V''$. T^{i+1}_{rl} is obtained from the spanning tree T^i_{rl} of G_i by adding the right leg $rl(u)$ to T^i_{rl}, and the right foot $rf(u)$ of u is a vertex in T^i_{rl}. Therefore T^{i+1}_{rl} is a spanning tree of G_{i+1}, as illustrated in Fig. 8 (a) where T^{i+1}_{rl} is drawn by thick lines.

Case 2: W_{i+1} satisfies condition (b). (See Fig. 6.)

Let $W_{i+1} = \{u_1, u_2, \cdots, u_h\}$, where $u_1, u_2, \cdots, u_h$ consecutively appear clockwise on $C_o(G_{i+1})$ in this order. Let $u_{h+1} = rf(u_h)$, then u_{h+1} is a vertex in T^i_{rl}. T^{i+1}_{rl} is obtained from T^i_{rl} by adding the h edges $(u_1, u_2), (u_2, u_3), \cdots, (u_h, u_{h+1})$ to T^i_{rl}. Thus T^{i+1}_{rl} is a spanning tree of G_{i+1} as illustrated in Fig. 8 (b).

Case 3: W_{i+1} satisfies condition (c). (See Fig. 7.)

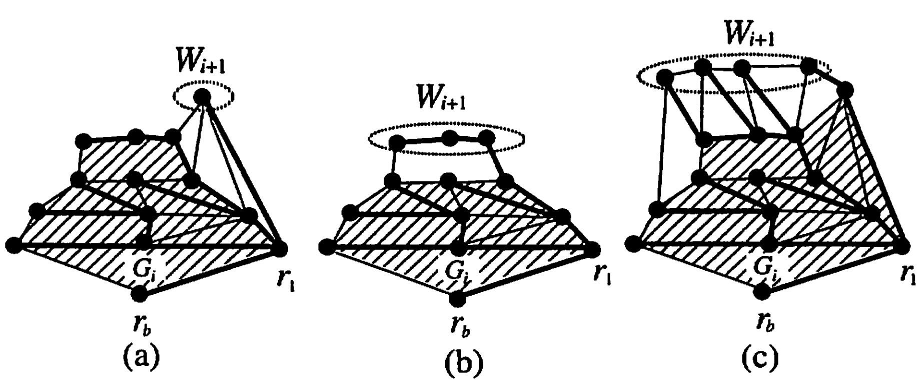

Fig. 8. Illustration for Lemma 3.

Let $W_{i+1} = \{u_1, u_2, \cdots, u_h\}$, where $u_1, u_2, \cdots, u_h$ consecutively appear clockwise on $C_o(G_{i+1})$ in this order. Then T_{rl}^{i+1} is obtained from T_{rl}^i by adding the right legs $rl(u_1), rl(u_2), \cdots, rl(u_h)$ to T_{rl}^i, and all the right feet $rf(u_1), rf(u_2), \cdots, rf(u_h)$ are vertices in T_{rl}^i. Therefore T_{rl}^{i+1} is a spanning tree of G_{i+1} as illustrated in Fig. 8 (c). □

We then have the following lemma.

Lemma 4 T_{rl}, T_{ll}, T_{la} and T_{ra} are spanning trees of G if both r_b and r_t are regarded as r.

Proof. By Lemma 3 T_{rl}^{m-1} is a spanning tree of G_{m-1}. Clearly $T_{rl} = T_{rl}^{m-1}$. Hence T_{rl} is a spanning tree of G if r_b in G_{m-1} is regarded as r in G.

Similarly, T_{ll}, T_{la} and T_{ra} are spanning trees of G. □

Thus it suffices to prove that the four spanning trees T_{rl}, T_{ll}, T_{la} and T_{ra} are *independent*. We first prove that any pair of an arm path and a leg path are internally disjoint. More precisely we have the following lemma.

Lemma 5 For any vertex $v \in V'' - \{r_b, r_t\}$, each of the four pairs $(P_{rl}(v), P_{la}(v))$, $(P_{rl}(v), P_{ra}(v))$, $(P_{ll}(v), P_{la}(v))$, and $(P_{ll}(v), P_{ra}(v))$ of paths are internally disjoint if both r_b and r_t are regarded as r.

Proof. We prove only that paths $P_{rl}(v)$ and $P_{la}(v)$ are internally disjoint in G'' if both r_b and r_t are regarded as r. Let $P_{rl}(v) = v, u_1, u_2, \cdots, u_j$, and $P_{la}(v) = v, w_1, w_2, \cdots, w_{j'}$, where $u_j = r_b$ and $w_{j'} = r_t$. By Lemma 2 both $P_{rl}(v)$ and $P_{la}(v)$ are simple paths. By (S1) and (S2), we have $rank(u_j) \leq rank(u_{j-1}) \leq \cdots \leq rank(u_1) \leq rank(v) \leq rank(w_1) \leq rank(w_2) \leq \cdots \leq rank(w_{j'})$. We furthermore have $rank(u_1) < rank(w_1)$ by (S4). Therefore $P_{rl}(v)$ and $P_{la}(v)$ have no common intermediate vertex. Hence they are internally disjoint if both r_b and r_t are regarded as r. □

We then prove that both the pair of arm paths and the pair of leg paths are internally disjoint. We now have the following lemma.

Lemma 6 Let u be any vertex in G'' other than r_b and r_t. Let $N(u) = \{v_1, v_2, \cdots, v_{d(u)}\}$, where $v_1 = lf(u)$. Let $v_s = rf(u)$, where $4 \leq s \leq d(u)$. (See Fig. 9.) If $rf(v_j) = lf(v_{j'}) = u$ for two distinct indices j and j', $1 \leq j, j' \leq s$, then $j < j'$.

Proof. Assume, on the contrary, that $rf(v_j) = lf(v_{j'}) = u$, $1 \leq j, j' \leq s$, but $j > j'$. Since neither r_b nor r_t has any leg, we have $v_j, v_{j'} \neq r_b, r_t$. Let $k = rank(v_j)$ and $k' = rank(v_{j'})$, then

$$rank(v_1), rank(v_s) \leq rank(u) \leq k, k'$$

and

$$2 \leq k, k' \leq m - 1.$$

There are the following three cases to consider.

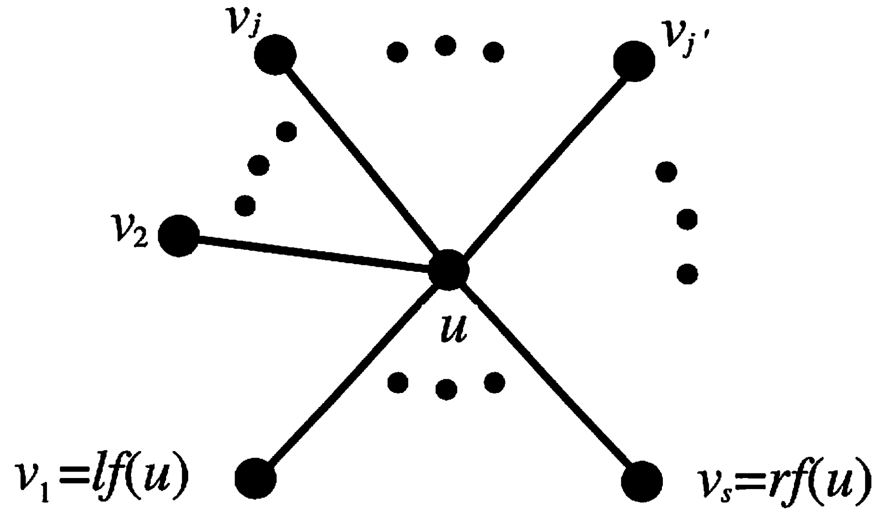

Fig. 9. Illustration for Lemma 6.

Case 1: $k = k'$.

In this case, $v_j, v_{j'} \in W_k$ and hence $|W_k| \geq 2$. Therefore W_k satisfies either condition (b) or (c). Consider first the case where W_k satisfies condition (b). (See Fig. 6.) Then, since $j > j'$ and $rf(v_j) = lf(v_{j'}) = u$, one can easily know that $rank(u) < k$, $u \notin W_k$, $v_{j'}$ is the first vertex in W_k, and v_j is the last vertex in W_k. Thus u is a cut vertex of G_k and hence G_k is not biconnected, contradicting the definition of the 4-canonical decomposition. Consider next the case where W_k satisfies condition (c). Then $u, v_{j'}, v_{j'+1}, \cdots, v_j, u$ is a simple cycle in G; the proper outside of the cycle contains root r; and the proper inside contains at least one vertex of G, say $lf(v_j) \neq r$. Thus the removal of three vertices $u, v_{j'}$ and v_j disconnects G, contradicting the 4-connectivity of G.

Case 2: $k < k'$.

Since $rank(v_j) = k$, v_j is on $C_o(G_k)$. However, $rl(v_j) = (v_j, u)$ is not always on $C_o(G_k)$. Consider first the case where (v_j, u) is not on $C_o(G_k)$. Then W_k must satisfy condition (c) and v_j must not be the last vertex in W_k; otherwise, $rl(v_j)$ would be on $C_o(G_k)$. Since (v_j, u) is not on $C_o(G_k)$, u is in the proper inside of $C_o(G_k)$. Since $k < k'$, $G_{k'}$ is obtained from G_k by adding the vertices in $W_{k+1} \bigcup W_{k+2} \bigcup \cdots \bigcup W_{k'}$ to G_k. All these added vertices are outside $C_o(G_k)$ in the plane graph $G_{k'}$. In particular, the vertex $v_{j'} \in W_{k'}$ is not on $C_o(G_k)$ but on $C_o(G_{k'})$. Therefore $v_{j'}$ cannot have u in the proper inside of $C_o(G_k)$ as a foot, contradicting $lf(v_{j'}) = u$. Consider next the case where $rl(v_j) = (v_j, u)$ is on $C_o(G_k)$. All neighbors of v_j clockwise appearing around v_j between $rf(v_j) = u$ and $lf(v_j)$ have ranks $\leq k$ and are contained in G_k. Therefore v_j precedes u on $C_o(G_k)$. Since $2 \leq k$, G_k is biconnected and hence $C_o(G_k)$ is a simple cycle. Since $rank(v_1), rank(v_s) \leq k$, v_1 and v_s are contained in G_k. Let v_t be the vertex succeeding u on $C_o(G_k)$, then $j + 1 \leq t \leq s$, and v_j, u, v_t consecutively appear clockwise on the simple cycle $C_o(G_k)$ in this order. Therefore, the edge $(v_{j'}, u)$ added to G_k must appear around u between (v_j, u) and (v_t, u) in $G_{k'}$ and hence in G''. Thus $v_j, v_{j'}, v_t$ must appear in $N(u)$ in this order. Therefore $j < j'$, contradicting the assumption that $j > j'$.

Case 3: $k > k'$.

Consider first the case where $ll(v_{j'}) = (v_{j'}, u)$ is not on $C_o(G_{k'})$. Then $W_{k'}$ must satisfy condition (c) and $v_{j'}$ must not be the first vertex in $W_{k'}$. Since $(v_{j'}, u)$ is not on $C_o(G_{k'})$, u is in the proper inside of $C_o(G_{k'})$. Since $k > k'$, G_k is obtained from $G_{k'}$ by adding the vertices in $W_{k'+1} \bigcup W_{k'+2} \bigcup \cdots \bigcup W_k$ to $G_{k'}$. Therefore $v_j \in W_k$ is not on $C_o(G_{k'})$ but on $C_o(G_k)$, and hence v_j cannot have u as a foot, contradicting $rf(v_j) = u$. Consider next the case where $ll(v_{j'}) = (v_{j'}, u)$ is on $C_o(G_{k'})$. Then $v_{j'}$ succeeds u on $C_o(G_{k'})$. Let v_t be the vertex preceding u on $C_o(G_{k'})$, then $1 \le t \le j' - 1$, and $v_t, v_j, v_{j'}$ appear in $N(u)$ in this order. Therefore $j < j'$, contradicting the assumption that $j > j'$. □

We then have the following lemma.

Lemma 7 Let v be any vertex in G'' other than r_b and r_t. Then both the pair of leg paths $P_{rl}(v)$ and $P_{ll}(v)$ and the pair of arm paths $P_{la}(v)$ and $P_{ra}(v)$ are internally disjoint.

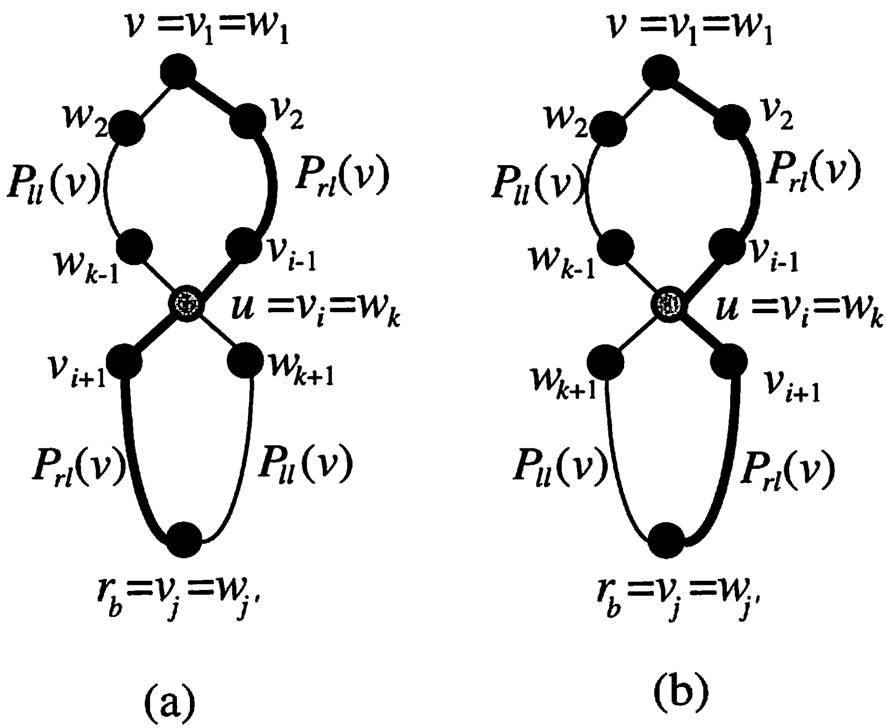

Fig. 10. Illustration for the proof of Lemma 7.

Proof. We prove only that $P_{rl}(v)$ and $P_{ll}(v)$ are internally disjoint. Suppose for a contradiction that $P_{rl}(v)$ and $P_{ll}(v)$ share an intermediate vertex. Let $P_{rl}(v) = v_1, v_2, \cdots, v_j$ and $P_{ll}(v) = w_1, w_2, \cdots, w_{j'}$, where $v = v_1 = w_1$ and $r_b = v_j = w_{j'}$. In Fig. 10 $P_{rl}(v)$ is drawn by thick line and $P_{ll}(v)$ by thin lines. Let u be the first intermediate vertex shared by the two paths $P_{rl}(v)$ and $P_{ll}(v)$ starting from v. Then $u = v_i = w_k$ for some i, $2 \leq i \leq j-1$, and k, $2 \leq k \leq j' - 1$, and vertices $v_2, v_3, \cdots v_{i-1}, w_2, w_3, \cdots, w_{k-1}$ are distinct with each other. Furthermore, either $w_{k-1}, v_{i-1}, w_{k+1}, v_{i+1}$ appear clockwise around u in this order as illustrated in Fig. 10 (a), or $w_{k-1}, v_{i-1}, v_{i+1}, w_{k+1}$ appear clockwise around u in this order as illustrated in Fig. 10 (b). If $w_{k-1}, v_{i-1}, w_{k+1}, v_{i+1}$ appear clockwise around u in this order, then $w_{k+1} = lf(u), v_{i+1} = rf(u), lh(u), rh(u)$ must appear clockwise around u in this order. This is a contradiction, because by the property (S3) $lf(u), lh(u), rh(u)$ and $rf(u)$ appear clockwise around v in this order. Thus one may assume that $w_{k-1}, v_{i-1}, v_{i+1}, w_{k+1}$ appear clockwise around u in this order. Then $w_{k+1} = lf(u), w_{k-1}, v_{i-1}, v_{i+1} = rf(u)$ appear clockwise around u in this order. However, $rf(v_{i-1}) = lf(w_{k-1}) = u$, contradicting Lemma 6. □

By Lemmas 4, 5 and 7 we have the following lemma.

Lemma 8 T_{rl}, T_{ll}, T_{la} and T_{ra} are four independent spanning trees of G rooted at r.

A plane embedding of G can be constructed in linear time [12]. A 4-canonical decomposition Π can be found in linear time [13]. T_{rl}, T_{ll}, T_{la} and T_{ra} can be found in linear time. Thus the running time of Algorithm FourTrees is $O(n)$. Thus we have the following theorem.

Theorem 1 Four independent spanning trees rooted at any designated vertex can be found in linear time for any 4-connected plane graph.

5. Conclusion

In this paper we give a linear-time algorithm to find four independent spanning trees of a 4-connected planar graph rooted at any designated vertex. Our algorithm is faster than the best known $O(n^3)$ algorithm in [7], and the complexity is optimal within a constant factor. Using four independent spanning trees, one can efficiently solve the 4-paths query problem for 4-connected planar graphs, which asks to find four internally disjoint paths connecting a specified pair of vertices. It is remained as a future work to find a linear-time algorithm for a larger class of graphs, say 4-connected graphs which are not always planar.

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