List Edge-Colorings of Series-Parallel Graphs

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SUMMARY Assume that each edge e of a graph G is assigned a list (set) L(e) of colors. Then an edge-coloring of G is called an L-edge-coloring if each edge e of G is colored with a color contained in L(e). In this paper, we prove that any series-parallel simple graph G has an L-edge-coloring if \( |L(e)| \geq \max\{3, d(v), d(w)\} \) for each edge e = vw, where d(v) and d(w) are the degrees of the ends v and w of e, respectively. Our proof yields a linear algorithm for finding an L-edge-coloring of series-parallel graphs.

key words: algorithm, list edge-coloring, series-parallel graph

1. Introduction

In this paper a graph means a finite “simple” graph without multiple edges and self-loops. We denote by \( G = (V, E) \) a graph with a vertex set V and an edge set E. We often denote V by \( V(G) \) and E by \( E(G) \). For each vertex \( v \in V \), we denote by \( d(v, G) \) or simply \( d(v) \) a degree of v, that is, the number of edges incident to v. We denote by \( \Delta(G) \) or simply \( \Delta \) a maximum degree of G. A graph is series-parallel if it contains no subgraph isomorphic to a subdivision of a complete graph \( K_4 \) on four vertices [6], [9]. For example, the graph in Fig. 1(a) is series-parallel. A series-parallel graph represents a network obtained by repeating “series connection” and “parallel connection.”

An edge-coloring of a graph G is a coloring of all edges in G such that any two edges sharing a common end are colored with different colors. The minimum number of colors necessary for an edge-coloring of G is called the chromatic index of G, and is denoted by \( \chi'(G) \). Clearly \( \Delta(G) \leq \chi'(G) \) for any graph G. On the other hand, Vizing has proved that \( \chi'(G) \leq \Delta(G) + 1 \) for any simple graph G [10]. The edge-coloring problem, asking whether a given graph G satisfies either \( \chi'(G) = \Delta(G) \) or not, is NP-complete [5]. On the other hand, it is known that \( \chi'(G) = \Delta(G) \) for some classes of graphs G; for example, \( \chi'(G) = \Delta(G) \) for any bipartite graph G [2], [7]; and \( \chi'(G) = \Delta(G) \) for any series-parallel graph G other than odd cycles [8], [9].

In this paper we deal with a generalized type of an edge-coloring called a “list edge-coloring,” which has some applications to a scheduling problem such as timetabling, routing in optical networks, and frequency assignment in cellular networks. Suppose that a set \( L(e) \) of colors, called a list of e, is assigned to each edge e of a graph G. Then an edge-coloring \( \psi \) of G is called an L-edge-coloring of G if \( \psi(e) \in L(e) \) for every edge e of G, where \( \psi(e) \) is a color assigned to e by \( \psi \). Figure 1(b) illustrates an L-edge-coloring of the graph G in Fig. 1(a); a list \( L(e) \) is attached to an edge e and a color \( \psi(e) \) is written in a box in Fig. 1(b). An ordinary edge-coloring is an L-edge-coloring for which \( L(e) \) is the same for all edges e, and hence an L-edge-coloring is a generalization of an ordinary edge-coloring. It is conjectured that any graph G has an L-edge-coloring if \( |L(e)| \geq \chi'(G) \) for each edge e [2], but this “list-coloring conjecture” has not been proved. On the other hand, Galvin proved that any bipartite graph G has an L-edge-coloring if \( |L(e)| \geq \Delta(G) \) for each edge e [4], and Borodin et al. improved the result by proving that any bipartite graph G has an L-edge-coloring if \( |L(e)| \geq \max\{d(v), d(w)\} \) for each edge e = vw [1]. Ju
den et al. proved that any series-parallel graph G has an L-edge-coloring if \( \Delta(G) \geq 3 \) and \( |L(e)| \geq \Delta(G) \) for each edge e [6]. Wu [11] improved the result by proving...
that any series-parallel graph \( G \) has an \( L \)-edge-coloring if either (i) or (ii) holds:

(i) \( |L(e)| \geq \max\{4, d(v), d(w)\} \) for each edge \( e = vw \); and

(ii) \( \Delta(G) \leq 3 \) and \( |L(e)| \geq 3 \) for each edge \( e \).

Fujino et al. gave a linear-time algorithm for finding such an \( L \)-edge-coloring of series-parallel graphs [3].

In this paper we prove that any series-parallel graph \( G \) has an \( L \)-edge-coloring if \( |L(e)| \geq \max\{3, d(v), d(w)\} \) for each edge \( e = vw \). Our result implies the results of Juvan et al. [6] and Wu [11] above. For any odd cycle \( C_n \), there is a list \( L \) such that \( C_n \) has no \( L \)-edge-coloring even if \( |L(vw)| \geq \max\{2, d(v), d(w)\} \) for each edge \( vw \in E(C_n) \). For any star \( K_{1,n-1}, n \geq 5 \), there is a list \( L \) such that \( K_{1,n-1} \) has no \( L \)-edge-coloring even if \( |L(vw)| \geq \max\{3, d(v) - 1, d(w)\} \) for each edge \( vw \in E(K_{1,n-1}) \). Thus our result is best-possible. Although our result is a slight modification of Wu’s [11], the proof for our result is much more involved than those in [6] and [11]. Since our proof is constructive, it immediately yields an efficient recursive algorithm to find an \( L \)-edge-coloring of a given series-parallel graph \( G \) if our condition holds. It takes time \( O(\min\{n\Delta, \ell\}) \) and hence runs in linear time where \( n \) is the number of vertices in \( G \) and \( \ell = \sum_{e \in E} |L(e)| \).

The rest of the paper is organized as follows. Section 2 gives definitions and two lemmas. Section 3 gives a proof of the main theorem using these lemmas. Finally, Sect. 4 gives conclusions.

2. Preliminaries

In this section, we present notations and lemmas which will be used in this paper.

The size of a graph \( G = (V, E) \) is \( |V| + |E| \). We say that a graph \( G' \) is smaller than a graph \( G \) if the size of \( G' \) is smaller than that of \( G \). An edge joining vertices \( v \) and \( w \) is denoted by \( vw \). We denote by \( G - v \) the graph obtained from a graph \( G \) by deleting a vertex \( v \in V \) and all edges incident to \( v \), and by \( G - e \) the graph obtained from \( G \) by deleting an edge \( e \). A graph obtained from \( G \) by deleting or contracting some edges is called a minor of \( G \) [2]. We denote by \( N_G(v) \) or simply \( N(v) \) the set of neighbors of \( v \) in \( G \). Since \( G \) is a simple graph, we have \( |N(v)| = d(v) \).

We have the following Lemma 1.

Lemma 1: Every non-empty series-parallel graph \( G \) satisfies one of the following ten conditions (a)–(j) (see Fig. 2):

(a) there is a vertex \( v \) of degree zero or one;

(b) there are two distinct vertices \( u \) and \( v \) of degree two such that \( N(u) = N(v) \);

(c) there are five distinct vertices \( u_1, u_2, v_1, v_2, w \) and two vertex sets \( Z_1, Z_2 \) such that \( N(w) = \{u_1, u_2, v_1, v_2\} \), \( Z_1 \cap Z_2 = \emptyset \), \( |Z_i| \leq 1 \), \( Z_i \cap \{u_1, u_2, v_1, v_2, w\} = \emptyset \), \( N(u_i) \supseteq \{v_i, w\} \cup Z_i \), \( N(v_i) = \{u_i, v_i\} \cup Z_i \), and \( N(z_i) = \{u_i, v_i\} \) if \( z_i \in Z_i \) where \( i = 1, 2 \); (Since there do not necessarily exist vertices \( z_1 \) and \( z_2 \), Fig. 2(c) indeed represents four substructures.)

(d) there are two distinct vertices \( u \) and \( v \) which have degree two and are adjacent to each other;

(e) there are four distinct vertices \( u, v, w, \) and \( z \) such that \( N(u) = \{v, w, z\} \), \( N(v) = \{u, w\} \) and \( z \notin N(w) \);

(f) there are four distinct vertices \( u, v, w, \) and \( z \) such that \( N(u) = \{v, w, z\} \), \( N(v) = \{u, w\} \) and \( N(w) = \{u, v, z\} \);

(g) there are five distinct vertices \( t, u, v, w, \) and \( z \) such that \( N(u) = \{v, w, z\} \), \( N(v) = \{u, w\} \) and \( N(w) = \{t, u, v, z\} \);

(h) there are five distinct vertices \( t, u, v, w, \) and \( z \) such that \( N(u) = \{v, w, z\} \), \( N(v) = \{u, w\} \) and \( N(z) = \{t, u, w\} \);

(i) there are five distinct vertices \( t, u, v, w, \) and \( z \) such that \( N(u) = \{v, w, z\} \), \( N(v) = \{u, w\} \), \( N(w) = \{t, u, v, z\} \) and \( d(t) = 2 \); and

(j) there are six distinct vertices \( s, t, u, v, w, \) and \( z \) such that \( N(s) = \{t, z\} \), \( N(t) \supseteq \{s, z\} \), \( N(u) = \{v, w, z\} \), \( N(v) = \{u, w\} \), \( N(z) \supseteq \{s, t, u, w\} \) and
edges incident to
fore, for each edge
Proof. See Appendix.
In the remaining of this paper, we denote the sub-
structures of $G$ described in (a)-(j) of Lemma 1 simply
by (a), (b), ..., (j). We also give the following Lemma 2
on $L$-edge-colorings of the graphs in Fig. 3. We omit
the proof since it is elementary but partly lengthy.
Lemma 2: Let $H$ be any of the eight graphs illustrated
in Figs. 3(a)–(h), and let $L$ be a list of $H$ such that
$|L(e)|$ is no smaller than the number attached to $e$
in Fig. 3 for each edge $e \in E(H)$. Then $H$ has an
$L$-edge-coloring.

Let $L$ be a list of a graph $G$ such that $|L(vw)| \geq
\max\{3, d(v), d(w)\}$ for each edge $vw \in E(G)$. Let $G'$
be a subgraph of $G$, and let $L'$ be a list of $G'$ such
that $L'(e) = L(e)$ for each edge $e \in E(G')$. Clearly
d$(v, G) \geq d(v, G')$ for each vertex $v \in V(G')$. Therefore,
for each edge $vw \in E(G')$, we have

$$|L'(vw)| = |L(vw)| \geq \max\{3, d(v, G), d(w, G)\} \geq \max\{3, d(v, G'), d(w, G')\}.$$  (1)

Suppose that we have already obtained an $L'$-edge-
coloring $\psi'$ of $G'$, and we are going to extend $\psi'$
to an $L$-edge-coloring $\psi$ of $G$ without altering the colors
of edges in $G'$. For a vertex $v \in V(G)$, we denote by
$C(v, \psi')$ the set of all colors that $\psi'$ have assigned to
d Edges incident to $v$ in $G'$, that is,

$$C(v, \psi') = \{\psi'(vx) \mid vx \in E(G')\}.$$  (2)

Then

$$|C(v, \psi')| = \begin{cases} d(v, G') & \text{if } v \in V(G'); \\
0 & \text{if } v \in V(G) - V(G'). \end{cases}$$  (3)

For each edge $vw \in E(G) - E(G')$, which has not been
colored, let

$$L_{av}(vw, \psi') = L(vw) - (C(v, \psi') \cup C(w, \psi')).$$  (4)

Then $L_{av}(vw, \psi')$ is the set of all colors in $L(vw)$ available
for $vw$ when $\psi'$ is extended to $\psi$, and we have

$$|L_{av}(vw, \psi')| \geq |L(vw)| - |C(v, \psi')| - |C(w, \psi')|.$$  (5)

3. Main Theorem

In this section, we prove the following main theorem.

Theorem 1: Let $G$ be a series-parallel graph, and let $L$
be a list of $G$ such that

$$|L(vw)| \geq \max\{3, d(v, G), d(w, G)\}$$  (6)

for each edge $vw \in E(G)$. Then $G$ has an $L$-edge-
coloring.

The outline of our proof of Theorem 1 is as follows.
Suppose for a contradiction that there exists a series-
parallel graph $G$ which has no $L$-edge-coloring for a list
$L$ satisfying Eq. (6). Assume that $G$ is a series-parallel
graph whose size is the smallest among these series-
parallel graphs. Lemma 1 implies that $G$ has one of the
ten substructures (a)–(j) in Fig. 2. Let $G'$ be a minor
of $G$ obtained from $G$ by deleting or contracting some
edges in the substructure. Since $G$ is series-parallel, $G'$
is series-parallel, too. Let $L'$ be an appropriate list of
$G'$ satisfying Eq. (6) for each edge $e \in E(G')$. The
assumption above implies that $G'$ has an $L'$-edge-coloring
$\psi'$. We can derive a contradiction by extending $\psi'$
to an $L$-edge-coloring of $G$.

Before giving the detail of a proof of Theorem 1, we present the following two lemmas.

Lemma 3: Assume that $G$ is a series-parallel graph,
that there are four distinct vertices $u, v, w$ and $z$
in $G$ such that $N(u) = \{v, w, z\}$ and $N(v) = \{u, w\}$
as illustrated in Fig. 4(a), and that a list $L$ of $G$ satisfies
Eq. (6). Assume that $G' = G - v - wu$ as illustrated in
Fig. 4(b), that $L'$ is a list of $G'$ such that $L'(e) = L(e)$
for each edge $e \in E(G')$, and that $G'$ has an $L'$-edge-
coloring $\psi'$. Then $G$ has an $L$-edge-coloring if one of
the following five Eqs. (7)–(11) holds:

$$\psi'(uz) \in (L(vw) - C(w, \psi'));$$  (7)
$$\psi'(uz) \notin L(uw);$$  (8)
$$L(vw) - L(uw) - \{\psi'(uz)\} \neq \emptyset;$$  (9)
$$L(vw) - L(uw) - C(w, \psi') \neq \emptyset;$$  (10)
and
$$|L(vw) - C(w, \psi')| \geq 3.$$  (11)

Proof. The edge $uz$ does not necessarily exist in $G$ as
indicated by a thin dotted line in Fig. 4(a) although the
edge $uz$ does not exist in the substructure (e) in Fig. 2.
Case 3: Equation (9) holds.

Thus

For Eqs. (3), (5) and (6) we have

\[ |L_{uv}(uv, \psi')| \geq |L(uv)| - d(u, G') \]
\[ \geq d(u, G) - (d(u, G) - 2) = 2, \quad (12) \]
\[ |L_{uv}(uv, \psi')| \geq |L(uv)| - (d(u, G') + d(w, G')) \]
\[ \geq 1, \quad (13) \]

and

\[ |L_{vw}(vw, \psi')| \geq |L(vw)| - d(w, G') \geq 2. \quad (14) \]

By Eqs. (2) and (4) we have

\[ \psi'(uz) \notin L_{av}(uv, \psi'), \quad (15) \]

and

\[ \psi'(uz) \notin L_{av}(uv, \psi'). \quad (16) \]

Since one of Eqs. (7)–(11) holds, there are the following five cases to consider.

Case 1: Equation (7) holds.

In this case we have \( \psi'(uz) \in (L(vw) - C(u, \psi')) = L_{av}(vw, \psi') \). By Eqs. (12) and (13) there are two distinct colors \( \alpha \in L_{av}(uv, \psi') \) and \( \beta \in L_{av}(uv, \psi') \). By Eqs. (15) and (16) we have \( \psi'(uz) \neq \alpha \) and \( \psi'(uz) \neq \beta \). Thus G has the following L-edge-coloring

\[ \psi(e) = \begin{cases} 
\psi'(e) & \text{if } e \in E(G) - \{uv, uw, vw\}; \\
\psi'(uz) & \text{if } e = uw; \\
\alpha & \text{if } e = uw; \\
\beta & \text{if } e = uv. 
\end{cases} \]

Case 2: Equation (8) holds.

By Eqs. (13) and (14) there are two distinct colors \( \alpha \in L_{av}(uv, \psi') \) and \( \beta \in L_{av}(uv, \psi') \). By Eqs. (2), (4) and (8) \( |L_{av}(uv, \psi')| \geq 3 \) and there is a color \( \gamma \in L_{av}(uv, \psi') \) such that \( \gamma \notin \{\alpha, \beta, \psi'(uz)\} \). Thus G has the following L-edge-coloring

\[ \psi(e) = \begin{cases} 
\psi'(e) & \text{if } e \in E(G) - \{uv, uw, vw\}; \\
\alpha & \text{if } e = uw; \\
\beta & \text{if } e = uv; \\
\gamma & \text{if } e = vw. 
\end{cases} \]

Case 3: Equation (9) holds.

In this case we have \( L(uv) - L(vw) - C(u, \psi') \neq \emptyset \) since \( C(u, \psi') = \{\psi'(uz)\} \). Clearly \( L_{av}(uv, \psi') = L(uv) - C(u, \psi') \). We thus have \( L_{av}(uv, \psi') - L(vw) \neq \emptyset \) and hence there is a color \( \alpha \in (L_{av}(uv, \psi') - L(vw)) \). Obviously

\[ \alpha \notin L(vw). \quad (17) \]

By Eq. (13) there is a color \( \beta \in L_{av}(uv, \psi') \). We then have the following two cases to consider.

Subcase 3(a): \( \alpha = \beta \).

By Eq. (12) there is a color \( \gamma \in L_{av}(uv, \psi') \) such that \( \gamma \neq \alpha \), and by Eq. (14) there is a color \( \tau \in L_{av}(vw, \psi') \) such that \( \tau \neq \gamma \). By Eq. (17) we have \( \alpha \neq \tau \in L_{av}(vw, \psi') \subseteq L(vw) \). Thus G has the following L-edge-coloring

\[ \psi(e) = \begin{cases} 
\psi'(e) & \text{if } e \in E(G) - \{uv, uw, vw\}; \\
\alpha & \text{if } e = uw; \\
\gamma & \text{if } e = uv; \\
\tau & \text{if } e = vw. 
\end{cases} \]

Subcase 3(b): \( \alpha \neq \beta \).

By Eq. (14) there is a color \( \gamma \in L_{av}(vw, \psi') \) such that \( \gamma \neq \beta \). By Eq. (17) we have \( \alpha \neq \gamma \in L_{av}(vw, \psi') \subseteq L(vw) \). Thus G has the following L-edge-coloring

\[ \psi(e) = \begin{cases} 
\psi'(e) & \text{if } e \in E(G) - \{uv, uw, vw\}; \\
\alpha & \text{if } e = uv; \\
\beta & \text{if } e = uw; \\
\gamma & \text{if } e = vw. 
\end{cases} \]

Case 4: Equation (10) holds.

Similarly as in Case 3, one can prove that G has an L-edge-coloring.

Case 5: Equation (11) holds.

By Eqs. (12) and (13) there are two distinct colors \( \alpha \in L_{av}(uv, \psi') \) and \( \beta \in L_{av}(uv, \psi') \). Since \( |L_{av}(vw, \psi')| = |L(vw) - C(u, \psi')| \geq 3 \) by Eqs. (4) and (11), there is a color \( \gamma \in L_{av}(vw, \psi') \) such that \( \gamma \notin \{\alpha, \beta\} \). Thus G has the following L-edge-coloring

\[ \psi(e) = \begin{cases} 
\psi'(e) & \text{if } e \in E(G) - \{uv, uw, vw\}; \\
\alpha & \text{if } e = uw; \\
\beta & \text{if } e = uv; \\
\gamma & \text{if } e = vw. 
\end{cases} \]

\[ \square \]

Lemma 4: Assume that G is a series-parallel graph, that there are four distinct vertices \( u, v, w, z \) such that \( N_G(u) = \{v, w, z\}, N_G(v) = \{u, w\} \) and \( d(w, G) \geq 4 \) as illustrated in Fig. 4(a), and that a list \( L \) of G satisfies \( |L(xy)| = \max \{3, d(x, G), d(y, G)\} \) for each edge \( xy \in E(G) \). Suppose that any series-parallel graph \( G' \) smaller than \( G \) has an \( L \)-edge-coloring for any list \( L' \) of \( G' \) satisfying Eq. (6). Then \( G \) has an \( L \)-edge-coloring if either

\[ L(uv) - L(vw) \neq \emptyset \quad (18) \]

or

\[ L(uw) \neq L(vw). \quad (19) \]
Proof. The assumption implies that

$$|L(uv)| = \max\{3, d(u, G), d(v, G)\} = 3,$$

(20)

$$|L(uw)| = \max\{3, d(u, G), d(w, G)\} = d(w, G),$$

(21)

and

$$|L(vw)| = \max\{3, d(v, G), d(w, G)\} = d(w, G).$$

(22)

We have the following two cases to consider.

Case 1: Equation (18) holds.

By Eq. (18) there is a color $\alpha \in L(uw) - L(vw)$. Let $G_1 = G - v$, and let $L_1$ be a list of $G_1$ such that

$$L_1(e) = \begin{cases} 
L(e) & \text{if } e \in E(G_1) - \{uw\}; \\
(L(vw) - L(uw)) \cup \{\alpha\} & \text{if } e = uw.
\end{cases}$$

(23)

By Eqs. (18) and (20) we have

$$|L(vw) \cap L(uw)| \leq 2.$$  

(24)

Since $\alpha \notin L(vw)$ and $d(w, G) \geq 4$, by Eqs. (22), (23) and (24) we have

$$|L(uw)| = |(L(vw) - L(uw)) \cup \{\alpha\}| = |L(vw)| - |L(uw) \cap L(vw)| + |\{\alpha\}| 
\geq d(w, G) - 1 
= \max\{3, d(u, G_1), d(w, G_1)\}. $$

(25)

One can know from Eqs. (1), (23) and (25) that the list $L_1$ of $G_1$ satisfies Eq. (6). Thus, by the supposition, $G_1$ has an $L_1$-edge-coloring $\psi_1$ since $G_1$ is a series-parallel graph smaller than $G$.

Let $G' = G - v - uw$, let $L'$ be a list of $G'$ such that $L'(e) = L(e)$ for each edge $e \in E(G')$, and let $\psi'$ be an $L'$-edge-coloring such that $\psi'(e) = \psi_1(e)$ for each edge $e \in E(G')$. Since $\psi_1(uw) \in L_1(uw) = (L(vw) - L(uw)) \cup \{\alpha\}$, either $\psi_1(uw) \in (L(vw) - L(uw))$ or $\psi_1(uw) = \alpha$.

Consider first the case where $\psi_1(uw) \in (L(vw) - L(uw))$. Since $\psi_1(uw) \notin C(w, \psi')$, in this case we have $\psi_1(uw) \in L(vw) - L(uw) - C(w, \psi')$, and hence Eq. (10) holds. Thus $G$ has an $L$-edge-coloring by Lemma 3.

Consider next the case where $\psi_1(uw) = \alpha$. Since $\alpha \in L(uw) - L(vw)$ and $\psi'(uz) = \psi_1(uz) = \alpha$, we have $\alpha \in L(vw) - L(uw) - \{\psi'(uz)\}$ and hence Eq. (9) holds. Thus $G$ has an $L$-edge-coloring by Lemma 3.

Case 2: Equation (18) doesn’t hold.

In this case we have

$$L(uw) \subseteq L(vw)$$

(26)

and we may assume that Eq. (19) holds, because either Eq. (18) or (19) holds. By Eqs. (21) and (22) we have $|L(uw)| = |L(vw)|$, and hence by Eq. (19) there is a color $\beta \in L(uw) - L(vw)$.

Let $G'$ be a graph such that $G' = G - v - uw$, and let $L'$ be a list such that $L'(e) = L(e)$ for each edge $e \in E(G')$. By Eq. (1) $L'$ satisfies Eq. (6). Since $G'$ is a series-parallel graph smaller than $G$, by the supposition $G'$ has an $L'$-edge-coloring $\psi'$. We then have the following two cases to consider.

Subcase 2(a): $\beta \in C(w, \psi')$.

Since $\beta \notin L(vw)$ and $|C(w, \psi')| = d(w, G) - 2$, by Eqs. (2), (3) and (22) we have

$$|L(vw) - C(w, \psi')| = |L(vw) - (C(w, \psi') - \{\beta\})| 
\geq |L(vw)| - |C(w, \psi') - \{\beta\}| 
= d(w, G) - |C(w, \psi')| - 1 
= 3$$

and hence Eq. (11) holds. Thus $G$ has an $L$-edge-coloring by Lemma 3.

Subcase 2(b): $\beta \notin C(w, \psi')$.

In this case we may assume that $\psi'(uz) \neq \beta$, as follows. Suppose $\psi'(uz) = \beta$. Then by the definition of $\beta$ and Eq. (26) we have $\psi'(uz) = \beta \notin L(uw) \subseteq L(vw)$, and hence Eq. (8) holds. Therefore $G$ has an $L$-edge-coloring by Lemma 3.

Since $\beta \in L(uw)$, $\beta \notin \psi'(uz)$, $\beta \notin C(w, \psi')$ and $L_{av}(uw, \psi') = (L(uw) - (C(w, \psi') \cup \{\psi'(uz)\}))$, we have $\beta \in L_{av}(uw, \psi')$. By Eqs. (3), (5), (20) and (22) we have

$$\lambda_{av}(uw, \psi') \geq d(u, G) - d(u, G') = 2$$

(27)

and

$$\lambda_{av}(vw, \psi') \geq d(w, G) - d(w, G') = 2.$$  

(28)

By Eqs. (27) and (28) there are two distinct colors $\gamma \in L_{av}(uw, \psi')$ and $\tau \in L_{av}(vw, \psi')$. Since $\beta \notin L(uw) \subseteq L(vw)$, we have $\beta \notin \{\gamma, \tau\}$. Thus $G$ has the following $L$-edge-coloring

$$\psi(e) = \begin{cases} 
\psi'(e) & \text{if } e \in E(G) - \{uw, vw, uv\}; \\
\beta & \text{if } e = uw; \\
\gamma & \text{if } e = uv; \\
\tau & \text{if } e = vw.
\end{cases}$$

We are now ready to give a proof of Theorem 1.

(Proof of Theorem 1)

Suppose for a contradiction that there exists a series-parallel graph $G$ which has no $L$-edge-coloring for a list $L$ satisfying Eq. (6). Assume that $G$ is the smallest one among all these series-parallel graphs. By Lemma 1 $G$ has one of the ten substructures (a)–(j) in Fig. 2, and hence we have the following three cases, Case A, Case B and Case C, to consider.
Case A: $G$ has a substructure other than (e) and (g).

We first assume that $G$ has a substructure (j). Let $H$ be a subgraph of $G$ induced by the edges $st, sz, tz, uw, uw, uz$ and $uv$. $H$ is drawn by thick solid lines in Fig.5(j), and is isomorphic to the graph in Fig.3(h). Let $G'$ be a graph obtained from $G$ by deleting these seven edges, and let $L'$ be a list of $G'$ such that $L'(e) = L(e)$ for each edge $e \in E(G')$. $G'$ is a series-parallel graph smaller than $G$, and $L'$ satisfies Eq. (6). Therefore $G'$ has an $L'$-edge-coloring $\psi'$. By Eqs. (3), (5) and (6) we have

$$|L_{av}(st, \psi')| \geq |L(st)| - d(t, G') \geq 3 - 1 = 2,$$
$$|L_{av}(sz, \psi')| \geq |L(sz)| - d(z, G') \geq d(z, G) - (d(z, G) - 3) = 3,$$
$$|L_{av}(tz, \psi')| \geq |L(tz)| - (d(t, G') + d(z, G')) \geq d(z, G) - (1 + (d(z, G) - 3)) = 2,$$
$$|L_{av}(uz, \psi')| \geq |L(uz)| - d(z, G') \geq d(z, G) - (d(z, G) - 3) = 3,$$
$$|L_{av}(uw, \psi')| \geq |L(uw)| \geq 3,$$
$$|L_{av}(wv, \psi')| \geq |L(wv)| - d(w, G') \geq d(w, G) - (d(w, G) - 2) = 2,$$

and

$$|L_{av}(vw, \psi')| \geq |L(vw)| - d(v, G') \geq d(w, G) - (d(w, G) - 2) = 2.$$

Hence by Lemma 2 $H$ has an $L_{av}$-edge-coloring $\psi_H$. Thus $G$ has an $L$-edge-coloring

$$\psi(e) = \begin{cases} \psi'_e(e) & \text{if } e \in E(G'); \\ \psi_H(e) & \text{if } e \in E(H). \end{cases}$$

This is a contradiction to the assumption that $G$ has no $L$-edge-coloring.

We next assume that $G$ has a substructure other than (e), (g) and (j). A proof for this case is similar as above. $G$ has a substructure (a), (b), (c), (d), (f), (h), or (i). Let $H$ be a subgraph drawn by thick solid lines in Figs. 5(a)–(i) for each case. Let $G'$ be a subgraph obtained from $G$ by deleting all edges in $H$, and let $L'$ be a list of $G'$ such that $L'(e) = L(e)$ for each edge $e \in E(G')$. Then $G'$ has an $L'$-edge-coloring $\psi'$. The number attached to each edge $e$ in Fig. 5 is the lower bound on $|L_{av}(e, \psi')|$. By Lemma 2 $H$ has an $L_{av}$-edge-coloring $\psi_H$. Thus $G$ has an $L$-edge-coloring

$$\psi(e) = \begin{cases} \psi'_e(e) & \text{if } e \in E(G'); \\ \psi_H(e) & \text{if } e \in E(H). \end{cases}$$

This is a contradiction.

Case B: $G$ has a substructure (e).

In this case there are four distinct vertices $u, v, w$ and $z$ in $G$ as illustrated in Fig. 2(e).

We may assume that $d(w, G) \geq 3$; if $d(w, G) = 2$, then $G$ would have a substructure (d), the case of which we have already treated in Case A.

We may also assume that $d(z, G) \geq 2$; if $d(z, G) = 1$ then $G$ would have a substructure (a).

The subgraph of $G$ drawn by thick lines in Fig. 2(e) is isomorphic to the graph in Fig. 3(f). Therefore one can easily show by Lemma 2 that $G$ would have an $L$-edge-coloring if $d(z, G) = 2$. We may thus assume

$$d(z, G) \geq 3. \tag{29}$$

Although each edge $e \in E(G)$ satisfies the inequality (6), we may assume without loss of generality that Eq. (6) holds in equality for each edge $e$, after eliminating some colors from each list $L(e)$. In particular we have

$$|L(uv)| = \max\{3, d(u, G), d(v, G)\} = 3, \tag{30}$$
$$|L(uw)| = \max\{3, d(u, G), d(w, G)\} = d(w, G), \tag{31}$$
$$|L(uz)| = \max\{3, d(u, G), d(z, G)\} = d(z, G), \tag{32}$$

and

$$|L(vw)| = \max\{3, d(v, G), d(w, G)\}$$
Fig. 6 Graphs $G$, $G' = G - v - uw$, and $G_2$.

Since $d(w, G) \geq 3$, either $d(w, G) = 3$ or $d(w, G) \geq 4$. We thus have the following two cases to consider.

Case 1: $d(w, G) = 3$.

Let $t$ be the vertex adjacent to $w$ other than $u$ and $v$. One may assume $t \neq z$; if $t = z$, then $G$ would have a substructure (I). By Eqs. (6) and (32) and (35) we have

$$|L(tw)| \geq \max\{3, d(t, G), d(w, G)\} \geq d(t, G). \quad (34)$$

Let $G_2$ be a graph obtained from $G - u - v$ by adding a new edge $wz$, as illustrated in Fig. 6(c). Since $G_2$ is a minor of a series-parallel graph $G$, $G_2$ is series-parallel, too. We then have the following two cases to consider.

Subcase 1(a): $L(uw) = L(vw)$.

Let $L_2$ be a list of $G_2$ such that

$$L_2(e) = \begin{cases} L(e) & \text{if } e \in E(G_2) - \{tw, wz\}; \\ L(uz) & \text{if } e = wz. \end{cases} \quad (35)$$

By Eqs. (6), (32) and (35) we have

$$|L_2(wz)| = |L(uz)| = \max\{3, d(u, G), d(z, G)\} \geq \max\{3, d(w, G_2), d(z, G_2)\} \quad (36)$$

and

$$|L_2(tw)| = |L(tw)| \geq \max\{3, d(t, G), d(w, G)\} \geq \max\{3, d(t, G_2), d(w, G_2)\}. \quad (37)$$

By Eqs. (1), (35), (36) and (37) $L_2$ satisfies Eq. (6) for each edge of $G_2$ and hence $G_2$ has an $L_2$-edge-coloring $\psi_2$.

Let $G' = G - v - uw$ as illustrated in Fig. 6(b), and let $L'$ be a list of $G'$ such that $L'(e) = L(e)$ for each edge $e \in E(G')$. Then $G'$ has an $L'$-edge-coloring $\psi'$.

$$\psi'(e) = \begin{cases} \psi_2(e) & \text{if } e \in E(G') - \{uz\}; \\ \psi_2(wz) & \text{if } e = uz. \end{cases}$$

We may assume that

$$\psi'(uz) \in L(uw); \quad (38)$$

if $\psi'(uz) \notin L(uw)$, then Eq. (8) holds and hence $G$ would have an $L$-edge-coloring by Lemma 3.

Since $C(w, \psi') = \{\psi'(tw)\}$, $L(uw) = L(vw)$ and $\psi'(uz) = \psi_2(wz) \neq \psi_2(tw) = \psi'(tw)$, by Eq. (38) we have $\psi'(uz) \in L(uw) - C(w, \psi') = L(vw) - C(w, \psi')$. Thus Eq. (7) holds, and hence by Lemma 3 $G$ has an $L$-edge-coloring.

Subcase 1(b): $L(uw) \neq L(vw)$.

Since $d(w, G) = 3$, by Eqs. (30) and (33) we have $|L(uw)| = |L(vw)| = d(w, G) = 3$. Since $L(uw) \neq L(vw)$, there are two distinct colors $\alpha \in L(uw) - L(vw)$ and $\beta \in L(vw) - L(uw)$. Let $\nu$ be a new color which does not appear in $L$, and let $L_2$ be a list of $G_2$ such that

$$L_2(e) = \begin{cases} (L(uw) - \{\beta\}) \cup \{\nu\} & \text{if } e = tw; \\ (L(vw) - \{\alpha\}) \cup \{\nu\} & \text{if } e = wz. \end{cases} \quad (39)$$

By Eqs. (1), (41) and (40), $L_2$ satisfies Eq. (6) for each edge of $G_2$ and hence $G_2$ has an $L_2$-edge-coloring $\psi_2$.

If $\psi_2(tw) = \nu$, then $\psi_2(tw) \notin L(uw)$, $\psi_2(tw) \in C(t, \psi_2)$ and $|L(uw)| \geq d(t, G) = |C(t, \psi_2)|$, and hence there is a color $\gamma \in L(uw) - C(t, \psi_2)$. Similarly, if $\psi_2(wz) = \nu$, then there is a color $\tau \in L(uw) - C(z, \psi_2)$.

Let $G' = G - v - uw$, and let $L'$ be a list of $G'$ such that $L'(e) = L(e)$ for each edge $e \in E(G')$. Then $G'$ has an $L'$-edge-coloring

$$\psi'(e) = \begin{cases} \psi_2(e) & \text{if } e \in E(G') - \{uw\}; \\ \psi_2(tw) & \text{if } e = tw \text{ and } \psi_2(tw) \neq \nu; \\ \psi_2(wz) & \text{if } e = uz \text{ and } \psi_2(wz) \neq \nu; \\ \tau & \text{if } e = uz \text{ and } \psi_2(wz) = \nu. \end{cases}$$

Since $\psi_2$ is an edge-coloring of $G_2$, either $\psi_2(tw) \neq \nu$ or $\psi_2(wz) \neq \nu$. If $\psi_2(tw) \neq \nu$, then $\psi'(tw) = \psi_2(tw) \neq \beta$ since $\beta \notin L_2(tw)$. If $\psi_2(wz) \neq \nu$, then $\psi'(uz) = \psi_2(wz) \neq \alpha$ since $\alpha \notin L_2(wz)$. We hence have

$$\psi'(uz) \neq \alpha \text{ or } \psi'(tw) \neq \beta. \quad (42)$$

By the symmetry of $G$, one may assume that $\psi'(uz) \neq \alpha$. Then we have $\alpha \in L(uw) - L(vw) - \{\psi'(uz)\}$, and
hence Eq. (9) holds. Thus by Lemma 3 $G$ has an $L$-edge-coloring.

Case 2: $d(w, G) \geq 4$.

In this case by Lemma 4 we may assume that

\begin{align}
L(uv) \subseteq L(vw),
L(uw) = L(vw).
\end{align}

(43) (44)

We first claim that

\begin{align}
L(uw) \cap L(uz) \neq \emptyset.
\end{align}

(45)

Suppose that $L(uw) \cap L(uz) = \emptyset$. Let $G' = G - v - uw$ as illustrated in Fig. 7(b), and let $L'$ be a list of $G'$ such that $L'(e) = L(e)$ for each edge $e \in E(G')$. Then $G'$ has an $L'$-edge-coloring $\psi'$ by the minimality of $G$. Since $\psi'(uz) \in L'(uz) = L(uz)$ and $L(uw) \cap L(uz) = \emptyset$, we have $\psi'(uz) \notin L(uw)$, and hence Eq. (8) holds. Thus $G$ has an $L$-edge-coloring by Lemma 3.

We then have the following two cases to consider, according to the value of $|L(uz) \cup (L(vw) - L(uw))|$.

Subcase 2(a): $|L(uz) \cup (L(vw) - L(uw))| \geq d(w, G) - 1$.

Let $G_2$ be a graph obtained from $G - u - v$ by adding a new edge $wz$ as illustrated in Fig. 7(d). Since $G_2$ is a minor of a series-parallel graph $G$, $G_2$ is series-parallel, too. Let $L_2$ be a list of $G_2$ such that

\begin{align}
L_2(e) &= \begin{cases}
L(e) & \text{if } e \in E(G_2) - \{wz\}; \\
L(uz) \cup (L(vw) - L(uw)) & \text{if } e = wz.
\end{cases}
\end{align}

(46)

By Eq. (32) we have

\begin{align}
|L_2(wz)| &= |L(uz) \cup (L(vw) - L(uw))| \\
&\geq d(z, G).
\end{align}

(47)

From Eqs. (46), $|L(uz) \cup (L(vw) - L(uw))| \geq d(w, G) - 1$, $d(w, G) \geq 4$, and $d(w, G_2) = d(w, G) - 1$, we have

\begin{align}
|L_2(wz)| &\geq \max\{d(w, G) - 1, d(z, G)\} \\
&= \max\{d(w, G), d(z, G)\}.
\end{align}

(48)

By Eqs. (1) and (47) $L_2$ satisfies Eq. (6) for each edge $e \in E(G_2)$, and hence $G_2$ has an $L_2$-edge-coloring $\psi_2$.

Consider first the case where $\psi_2(wz) \in L_2(wz)$. Let $G' = G - v - uw$, and let $L'$ be a list of $G'$ such that $L'(e) = L(e)$ for each edge $e \in E(G')$. Then $G'$ has an $L'$-edge-coloring
a list of $G_1$ such that

$$L_1(e) = \begin{cases} L(e) & \text{if } e \in E(G_1) - \{uw, uz\}; \\ (L(vw) - (L(uw) - \{\alpha\})) \cup \{\nu\} & \text{if } e = uw; \\ (L(uz) - \{\alpha\}) \cup \{\nu\} & \text{if } e = uz. \end{cases} \quad (53)$$

By Eqs. (29), (30), (32), (33), (43), (44), (52) and (53) we have

$$|L_1(uw)| = |L(vw)| - |(L(uw) - \{\alpha\})| + |\{\nu\}| = d(w, G) - (3 - 1) + 1 = d(w, G) - 1 = \max\{3, d(u, G_1), d(w, G_1)\}. \quad (54)$$

and

$$|L_1(uz)| = |L(uz) - \{\alpha\}| + |\{\nu\}| = (d(z, G) - 1) + 1 = d(z, G) = \max\{3, d(u, G_1), d(z, G_1)\}. \quad (55)$$

By Eqs. (1), (54) and (55) $L_1$ satisfies Eq. (6) for each edge $e \in E(G_1)$, and hence $G_1$ has an $L_1$-edge-coloring $\psi_1$. Since $\psi_1(uz) \in L_1(uz) = (L(uz) - \{\alpha\}) \cup \{\nu\}$, either $\psi_1(uz) \in L(uz) - \{\alpha\}$ or $\psi_1(uz) = \nu$.

We now claim that $\psi_1(uz) = \nu$. Suppose that $\psi_1(uz) \neq \nu$. Then $\psi_1(uz) \in L(uz) - \{\alpha\}$, and hence by Eq. (51) we have

$$\psi_1(uz) \notin L(uw). \quad (56)$$

Let $G' = G - v - w$, and let $L'$ be a list of $G'$ such that $L'(e) = L(e)$ for each edge $e \in E(G')$. Then $G'$ has an $L'$-edge-coloring $\psi'$ such that $\psi'(e) = \psi_1(e)$ for each edge $e \in E(G')$. Since $\psi'(uz) = \psi_1(uz) \notin L(uw)$ by Eq. (56), Eq. (8) holds and hence $G$ has an $L$-edge-coloring by Lemma 3.

Since $\psi_1(uz) = \nu$ and $\alpha \in L(uz)$, by Eqs. (3) and (32) we have $|L(uz) - \{\alpha\}| = |C(z, \psi_1) - \{\nu\}| = d(z, G) - 1$.

We then claim that $L(uz) - \{\alpha\} = C(z, \psi_1) - \{\nu\}$. Suppose that $L(uz) - \{\alpha\} \neq C(z, \psi_1) - \{\nu\}$. Then there is a color $\beta \in (L(uz) - \{\alpha\}) - (C(z, \psi_1) - \{\nu\})$. Let $G' = G - v - w$, and let $L'$ be a list of $G'$ such that $L'(e) = L(e)$ for each edge $e \in E(G')$. Then $G'$ has an $L'$-edge-coloring

$$\psi'(e) = \begin{cases} \psi_1(e) & \text{if } e \in E(G') - \{uz\}; \\ \beta & \text{if } e = uz. \end{cases}$$

Since $\beta \in (L(uz) - \{\alpha\}$, by Eq. (51) we have $\psi'(uz) = \beta \notin L(uw)$ and hence Eq. (8) holds. Therefore by Lemma 3 $G$ has an $L$-edge-coloring.

Since $L(uz) - \{\alpha\} = C(z, \psi_1) - \{\nu\}$ and $\alpha \in L(uz)$, we have $\alpha \in L(uz) - (C(z, \psi_1) - \{\nu\})$. Let $G' = G - v - w$, and let $L'$ be a list of $G'$ such that $L'(e) = L(e)$ for each edge $e \in E(G')$. Then $G'$ has an $L'$-edge-coloring

$$\psi'(e) = \begin{cases} \psi_1(e) & \text{if } e \in E(G') - \{uz\}; \\ \alpha & \text{if } e = uz. \end{cases}$$

Since $\psi_1$ is an edge-coloring, $\psi_1(ww) \neq \psi_1(uz) = \nu$. By Eqs. (44) and (53) we have $\psi_1(ww) \in L_1(ww) = (L(uw) - (L(uw) - \{\alpha\})) \cup \{\nu\}$. We hence have $\psi_1(ww) \in L(uw) - L(ux) - \{\alpha\}$. Thus either $\psi_1(ww) = \alpha$ or $\psi_1(ww) \in L(uw) - L(uw)$.

Consider first the case where $\psi_1(ww) = \alpha$. Since $\psi_1(ww) \neq \psi_1(uz) = \nu$, by Eqs. (44) and (53) we have $\psi_1(ww) \in L_1(ww) - \{\nu\} \subseteq L(uw) = L(vw)$. Since $C(w, \psi') = C(w, \psi_1) - \{\psi_1(ww)\}$, we have $\psi_1(ww) \notin C(w, \psi')$ and hence $\psi'(uz) = \alpha = \psi_1(ww) \in L(uw) - C(w, \psi')$. Thus Eq. (7) holds, and hence by Lemma 3 $G$ has an $L$-edge-coloring.

Consider next the case where $\psi_1(ww) \in L(uw) - L(uw)$. In this case we have $\psi_1(ww) \in L(uw) - L(uw) - C(w, \psi')$ since $\psi_1(ww) \notin C(w, \psi') = C(w, \psi_1) - \{\psi_1(ww)\}$. Thus by Eq. (44), we have $\psi_1(ww) \in L(uw) - L(uw) - C(w, \psi')$, and hence Eq. (10) holds. Hence by Lemma 3 $G$ has an $L$-edge-coloring.

Case C: $G$ has a substructure (g).

In this case there are five distinct vertices $t, u, v, w$ and $z$ in $G$ as illustrated in Fig. 2(g) and Fig. 8(a).

We may assume that

$$d(z, G) \geq 4; \quad \text{if } d(z, G) = 2 \text{ then } G \text{ would have a substructure (b),} \quad \text{and if } d(z, G) = 3 \text{ then } G \text{ would have a substructure (h).} \quad (57)$$

$$\text{We may assume that}$$

$$|L(uw)| = \max\{3, d(u, G), d(v, G)\} = 3, \quad (58)$$

$$|L(uz)| = \max\{3, d(u, G), d(w, G)\} = 4, \quad (59)$$

$$|L(vw)| = \max\{3, d(v, G), d(w, G)\} = d(z, G), \quad (60)$$

and

$$|L(ww)| = \max\{3, d(v, G), d(w, G)\} = 4. \quad (61)$$

By Lemma 4 we may assume that

$$L(uw) \subseteq L(vw) \quad \text{and}$$

$$L(uw) = L(uw). \quad (62)$$

By Eqs. (58), (61) and (62) there is a color $\alpha$ such that $L(uw) - L(uw) = \{\alpha\}$.
Let $G_3 = G - u - v$ as illustrated in Fig. 8(b), and let $L_3$ be a list of $G_3$ such that
\[
L_3(e) = \begin{cases} L(e) & \text{if } e \in E(G_3) - \{wz\}; \\ L(wz) - \{\alpha\} & \text{if } e = wz. \end{cases}
\]
Then, since $|L(wz)| \geq \max\{3, d(w, G), d(z, G)\}$, by Eq. (57) we have
\[
|L_3(wz)| \geq |L(wz)| - 1 \geq \max\{3, d(w, G_3), d(z, G_3)\}. \quad (64)
\]
One can hence know that $L_3$ satisfies Eq. (6) for each edge of $G_3$. Thus $G_3$ has an $L_3$-edge-coloring $\psi_3$.

By Eqs. (3), (5), (58)–(61) we have
\[
|L_{av}(uw, \psi_3)| \geq |L(uw)| \geq 3, \quad (65)
\]
\[
|L_{av}(uw, \psi_3)| \geq |L(uw)| - d(w, G_3) \geq 4 - 2 = 2, \quad (66)
\]
\[
|L_{av}(uz, \psi_3)| \geq |L(uw)| - d(z, G_3) \geq d(z, G) - (d(z, G) - 1) = 1, \quad (67)
\]
and
\[
|L_{av}(vw, \psi_3)| \geq |L(vw)| - d(w, G_3) = 4 - 2 = 2. \quad (68)
\]

Since $\psi_3(wz) \in L_3(wz) = L(wz) - \{\alpha\}$, we have $\psi_3(wz) \notin \{\alpha\} = L(uw) - L(uw)$ and hence either $\psi_3(wz) \in L(uw)$ or $\psi_3(wz) \notin L(uw)$. Thus there are the following two cases to consider.

Case 1: $\psi_3(wz) \in L(uw)$.

Since $L_{av}(uw, \psi_3) = L(uw)$, we have $\psi_3(wz) \in L_{av}(uw, \psi_3) = L(wz)$. By Eq. (67) there is a color $\beta \in L_{av}(uz, \psi_3)$. By Eq. (66) there is a color $\gamma \in L_{av}(uw, \psi_3)$ such that $\gamma \neq \beta$. By Eq. (68) there is a color $\tau \in L_{av}(vw, \psi_3)$ such that $\tau \neq \gamma$. Since edge $wz$ is adjacent to the edges $uw, uw$ and $uz$ in $G$, we have $\psi_3(wz) \notin L_{av}(uw, \psi_3) \cup L_{av}(uw, \psi_3) \cup L_{av}(uz, \psi_3)$ and hence $\psi_3(wz) \notin \{\beta, \gamma, \tau\}$. Thus $G$ has an $L$-edge-coloring
\[
\psi(e) = \begin{cases} \psi_3(e) & \text{if } e \in E(G) - \{uw, uw, uz, vw\}; \\ \beta & \text{if } e = uz; \\ \gamma & \text{if } e = uw; \\ \tau & \text{if } e = vw. \end{cases}
\]

Case 2: $\psi_3(wz) \notin L(uw)$.

Since $\psi_3(wz) \notin L(uw)$, $d(w, G_3) = 2$ and $\psi_3(wz) \in C(w, \psi_3)$, we have
\[
|L(uw) \cap C(w, \psi_3)| = |L(uw) \cap (C(w, \psi_3) - \{\psi_3(wz)\})| \leq |C(w, \psi_3) - \{\psi_3(wz)\}| = |C(w, \psi_3)| - 1 = 1. \quad (69)
\]
By Eqs. (2), (4), (61), (69) and $\psi_3(wz) \notin L(uw)$ we have
\[
|L_{av}(uw, \psi_3)| = |L(uw) - C(w, \psi_3)| = |L(uw)| - |L(uw) \cap C(w, \psi_3)| \geq 4 - 1 = 3. \quad (70)
\]
By Eq. (67) there is a color $\beta_1 \in L_{av}(uz, \psi_3)$. By Eq. (66) there is a color $\beta_2 \in L_{av}(uw, \psi_3)$ such that $\beta_2 \neq \beta_1$. By Eq. (65) there is a color $\beta_3 \in L_{av}(uw, \psi_3)$ such that $\beta_3 \notin \{\beta_1, \beta_2\}$. By Eq. (70) there is a color $\beta_4 \in L_{av}(uw, \psi_3)$ such that $\beta_4 \notin \{\beta_2, \beta_3\}$. Thus $G$ has an $L$-edge-coloring
\[
\psi(e) = \begin{cases} \psi_3(e) & \text{if } e \in E(G) - \{uw, uw, uz, vw\}; \\ \beta_1 & \text{if } e = uz; \\ \beta_2 & \text{if } e = uw; \\ \beta_3 & \text{if } e = wz; \\ \beta_4 & \text{if } e = vw. \end{cases}
\]

\begin{proof}

\end{proof}

\section{Conclusions}

In this paper we proved Theorem 1, which implies the known results in [6], [11]. For any odd cycle $C_n$, there is a list $L$ such that $C_n$ has no $L$-edge-coloring even if $|L(uw)| \geq \max\{2, d(v), d(w)\}$ for each edge $vw \in E(C_n)$. For any star $K_{1, n-1}$, $n \geq 5$, there is a list $L$ such that $K_{1, n-1}$ has no $L$-edge-coloring even if $|L(uw)| \geq \max\{3, d(v) - 1, d(w)\}$ for each edge $vw \in E(K_{1, n-1})$. Thus Theorem 1 is best possible. Since our proof of Theorem 1 is constructive, it yields an efficient recursive algorithm to find an $L$-edge-coloring of a series-parallel graph $G$ if $L$ satisfies Eq. (6) for each edge of $G$. The algorithm takes time $O(\min\{n, \ell\})$ and hence runs in linear time, where $\ell = \Sigma_{e \in E(L)}|L(e)|$.

One of the remaining problems is to characterize a class of series-parallel graphs that have $L$-edge-colorings if
\[
|L(uw)| \geq \max\{d(v, G), d(w, G)\}
\]
for each edge $vw \in E(G)$.

\section*{References}


Appendix: Proof of Lemma 1

Juvan et al. proved the following lemma [6].

Lemma 5: Every non-empty series-parallel graph $G$ satisfies one of the following five conditions (a)–(e) (see Fig. A-1):

(a) there is a vertex $v$ of degree zero or one;
(b) there are two distinct vertices $u$ and $v$ of degree two and $N(u) = N(v)$;
(c) there are five distinct vertices $u_1, u_2, v_1, v_2$ and $w$ such that $N(v_1) = \{u_1, w\}$, $N(v_2) = \{u_2, w\}$ and $N(w) = \{u_1, u_2, v_1, v_2\}$;
(d) there are two distinct vertices $u$ and $v$ which have degree two and are adjacent to each other; and
(e) there are four distinct vertices $u, v, w, z$ such that $N(u) = \{v, w, z\}$ and $N(v) = \{u, w\}$. (There does not always exist the edge $wz$ in Fig. A-1(e), while there does not exist $wz$ in Fig. 2(e).)

Using Lemma 5 we give a proof of Lemma 1, in which (a)–(j) mean those in Lemma 1 unless specified otherwise. Note that the substructures in Figs. A-1(a), (b) and (d) are the same as those in Figs. 2(a), (b) and (d), respectively, and that the substructure in Fig. A-1(e) is the same as that in Fig. 2(c) of the case $Z_1 = Z_2 = \emptyset$.

Proof of Lemma 1. Suppose that there exists a non-empty series-parallel graph $G = (V, E)$ which satisfies none of (a)–(j) in Lemma 1, and we will derive a contradiction. Assume that $G$ is the smallest one among such series-parallel graphs. Since $G$ satisfies none of (a)–(d) in Lemma 1, $G$ satisfies none of (a)–(d) in Lemma 5. Then $G$ satisfies (e) in Lemma 5. Thus $G$ has four distinct vertices $u, v, w, z$ such that $N(u) = \{v, w, z\}$ and $N(v) = \{u, w\}$. Since $G$ does not satisfy (e) in Lemma 1, we have

$$wz \in E,$$  \hspace{1cm} (A-1)

and hence $d(w, G) \geq 3$ and $d(z, G) \geq 2$.

We have

$$d(w, G) \geq 5;$$  \hspace{1cm} (A-2)

if $d(w, G) = 3$ then $G$ would satisfy (f), while if $d(w, G) = 4$ then $G$ would satisfy (g).

We have

$$d(z, G) \geq 4;$$  \hspace{1cm} (A-3)

if $d(z, G) = 2$ then $G$ would satisfy (b), while $d(z, G) = 3$ then $G$ would satisfy (h).

We have

$$w$$ is not adjacent to any vertex of degree two other than $v$ in $G$; \hspace{1cm} (A-4)

otherwise, $G$ would satisfy (i).

Let $G' = G - u - v$. Then by Eqs. (A-2) and (A-3) we have

$$d(w, G') \geq 3,$$  \hspace{1cm} (A-5)

and

$$d(z, G') \geq 3.$$  \hspace{1cm} (A-6)

For any vertex $x \in V(G')$

$$N_{G'}(x) \neq N_G(x) \iff x \in \{w, z\}.$$  \hspace{1cm} (A-7)

By Eqs. (A-5), (A-6) and (A-7), any vertex $x \in V(G')$ with $d(x, G') \leq 2$ satisfies

$$N_G(x) = N_{G'}(x).$$  \hspace{1cm} (A-8)

By Eqs. (A-4) and (A-8)

$$w$$ is not adjacent to any vertex of degree two in $G'$. \hspace{1cm} (A-9)

By Eq. (A-1),

$$wz \in E'.$$  \hspace{1cm} (A-10)

Since $G'$ is a series-parallel graph smaller than $G$, $G'$ satisfies one of (a)–(j). We thus have the following four cases to consider.
Case 1: $G'$ satisfies (a), (b) or (d).

We assume that $G'$ satisfies (b). (A proof for the other case is similar.) Then let $V^b = \{u^b, v^b, w^b, z^b\}$ be a set of vertices in $G'$ satisfying (b). Since $d(u^b, G') = d(v^b, G') = 2$, by Eq. (A-8) we have $N_G(u^b) = N_G(v^b)$ and $N_G(w^b) = N_G(z^b)$. Thus $V^b$ also satisfies (b) in $G$, contrary to the assumption that $G$ does not satisfy (b).

Case 2: $G'$ satisfies (c).

In this case $G'$ has a vertex set $V^c = \{u^c_1, u^c_2, v^c_1, v^c_2, w^c\} \cup Z^1_2 \cup Z^2_2$ satisfying (c). We have either $w \in V^c - \{u^c_1, u^c_2\}$ or $z \in V^c - \{u^c_1, u^c_2\}$; otherwise, by Eq. (A-7) $N_G(x) = N_G(x)$ for each vertex $x \in V^c - \{u^c_1, u^c_2\}$, and hence $V^c$ would satisfy (c) in $G$, a contradiction.

We now claim that $w, z \in V^c$. Otherwise, either $w \in V^c - \{u^c_1, u^c_2\}$ and $z \notin V^c$ or $w \notin V^c$ and $z \in V^c - \{u^c_1, u^c_2\}$. In the former case, $z \in N_G(w) \subseteq V^e$, a contradiction. In the latter case, $w \in N_G(z) \subseteq V^e$, a contradiction.

We have the following three cases to consider.

Subcase 2(a): $|Z^1_2| + |Z^2_2| = 0$.

Only $u^c_1, u^c_2$ and $w^c$ among the vertices in $V^c$ may have degree three or more in $G'$. We hence have $w \in \{u^c_1, u^c_2, w^c\} \subseteq V^c$ since $w \in V^c$ and $d(w, G') \geq 3$ by Eq. (A-5). However, since each of the vertices $u^c_1, u^c_2$ and $w^c$ is adjacent to a vertex of degree two in $G'$, we have $w \notin \{u^c_1, u^c_2, w^c\}$ by Eq. (A-9), a contradiction.

Subcase 2(b): $|Z^1_2| + |Z^2_2| = 1$.

In this case we may assume without loss of generality that $Z^1_2 = \{z^1_2\}$ and $Z^2_2 = \emptyset$. Then only $u^c_1, u^c_2, v^c_1$ and $w^c$ among the vertices in $V^c$ may have degree three or more in $G'$. We hence have $w \in \{u^c_1, u^c_2, v^c_1, w^c\} \subseteq V^c$. However, since each of the vertices $u^c_1, u^c_2, v^c_1$ and $w^c$ is adjacent to a vertex of degree two in $G'$, we have $w \notin \{u^c_1, u^c_2, v^c_1, w^c\}$ by Eq. (A-9), a contradiction.

Subcase 2(c): $|Z^1_2| + |Z^2_2| = 2$.

In this case there are two distinct vertices $z^1_2$ and $z^2_2$ such that $Z^1_2 = \{z^1_2\}$ and $Z^2_2 = \{z^2_2\}$. Then only $u^c_1, u^c_2, v^c_1, v^c_2$ and $w^c$ among the vertices in $V^c$ may have degree three or more in $G'$. Therefore we hence have $w, z \in \{u^c_1, u^c_2, v^c_1, v^c_2, w^c\} \subseteq V^c$ since $d(w, G') \geq 3$ and $d(z, G') \geq 3$ by Eqs. (A-5) and (A-6). Vertices $u^c_1, u^c_2, v^c_1, v^c_2$ are adjacent to a vertex of degree two in $G'$, but $w$ is not so by Eq. (A-9). Therefore we have $w \notin \{u^c_1, u^c_2, v^c_1, v^c_2\}$. Thus we have $w = w^c$ and $z \in \{u^c_1, u^c_2, v^c_1, v^c_2\}$. If $z = v^c_i$, $i = 1, 2$, then the vertices $u, u^c_i, v, w, z$ and $z^1_2$ would satisfy (c) in $G$, a contradiction. If $z = u^c_i$, $i = 1, 2$, then the vertices $w, u, v^c_i, w, z$ and $z^1_2$ would satisfy (j) in $G$, a contradiction.

Case 3: $G'$ satisfies (e).

In this case $G'$ has a vertex set $V^e = \{v^e_1, v^e_2, w^e, z^e\}$ satisfying (e), and $w^e z^e \notin E(G')$.

We now claim that either $w \in \{v^e_1, v^e_2\}$ or $z \in \{v^e_1, v^e_2\}$. Otherwise, $w^e, v^e \notin \{w, z\}$, and hence by Eq. (A-7) $N_G(v^e) = N_G(w^e)$ and $N_G(v^e) = N_G(v^e)$. Since $w^e z^e \notin E(G')$ and $w^e z^e \notin E(G')$, we have $\{v^e_1, v^e_2\} \neq \{w, z\}$ and hence by Eq. (A-7) $w^e z^e \notin E(G)$. Thus $V^e$ would satisfy (e) in $G$, a contradiction.

We then claim $w, z \in V^e$. If $w \notin \{v^e_1, v^e_2\}$, then $N_G(w) \subseteq V^e$ and hence by Eq. (A-10) $z \notin V^e$. If $z \notin \{v^e_1, v^e_2\}$, then $N_G(z) \subseteq V^e$ and hence by Eq. (A-10) $w \notin V^e$.

By Eq. (A-5) $d(w, G') \geq 3$ and by Eq. (A-6) $d(z, G') \geq 3$. Only $w^e, v^e$ and $z^e$ among the vertices in $V^e$ may have degree three or more in $G'$. Therefore $w, z \in \{v^e_1, v^e_2, z^e\} \subseteq V^e$. Since each of vertices $w^e$, $v^e$, and $z^e$ is adjacent to a vertex of degree two in $G'$, by Eq. (A-9) we have $w \notin \{v^e_1, v^e_2\}$, and hence $w = z^e$. We also have $z = v^e_1$ since $z \in V^e$, $w^e \in E(G')$, and $N_G(z^e) \cap V^c = \{w^e\}$. Then $d(z, G') = 4$, $d(v, G) = d(v^e, G) = 2$, and hence the vertices $u, v, v^e, w^e$ and $z$ would satisfy (c) in $G$, a contradiction.

Case 4: $G'$ satisfies (f), (g), (h), (i) or (j).

One can derive a contradiction similarly as in Case 3. □