Edge-Coloring Algorithms

Shin-ichi Nakano, Xiao Zhou and Takao Nishizeki

Graduate School of Information Sciences

Tohoku University, Sendai 980-77, Japan

Abstract. The edge-coloring problem is one of the fundamental problems on graphs, which often appears in various scheduling problems like the file transfer problem on computer networks. In this paper, we survey recent advances and results on the classical edge-coloring problem as well as the generalized edge-coloring problems, called the f-coloring and fg-coloring problems. In particular we review various upper bounds on the minimum number of colors required to edge-color graphs, and present efficient algorithms to edge-color graphs with a number of colors not exceeding the upper bounds.

1 A survey of the edge-coloring problem

1.1 A history of the edge-coloring problem

The edge-coloring problem is one of the fundamental problems on graphs. A graph $G = (V, E)$ is an ordered pair of vertex set V and edge set E. An edge in E joins two vertices in V. Throughout the paper we let $n = |V|$ and $m = |E|$. The edge-coloring problem is to color all edges of a given graph with the minimum number of colors so that no two adjacent edges are assigned the same color. Fig. 1 illustrates an edge-coloring of a graph with four colors. A set of edges which are not adjacent each other is called a *matching.* Since each set of edges colored with the same color is a matching, an edge-coloring of a graph is indeed a partition of E to matchings.

We now historically review the edge-coloring problem. The edge-coloring problem was posed in 1880 in relation with the well-known four-color conjecture: every map could be colored with four colors so that any neighboring countries have different colors. It took more than 100 years to prove the conjecture affirmatively in 1976 with the help of computers since it was posed in 1852. The first paper that dealt with the edge-coloring problem was written by Tait in 1889 [10]. In the paper Tait proved that the four-color problem is equivalent with the problem of edge-coloring every planar 3-connected cubic graph with three colors. The minimum number of colors needed to edge-color G is called the *chromatic index* $\chi'(G)$ of G. The maximum degree of graph G is denoted by $\Delta(G)$ or simply by Δ. Obviously $\chi'(G) \geq \Delta(G)$ since all edges incident to the same vertex must be assigned different colors. König [20] proved that every bipartite graph can be edge-colored with exactly $\Delta(G)$ colors, that is $\chi'(G) = \Delta(G)$. Shannon [30] proved that every graph can be edge-colored with at most $3\Delta(G)/2$ colors, that is $\chi'(G) \leq 3\Delta(G)/2$. Vizing [32] proved that $\chi'(G) \leq \Delta(G) + 1$ for every simple graph. A few other upper bounds on $\chi'(G)$ have been known [2,14,17,27].

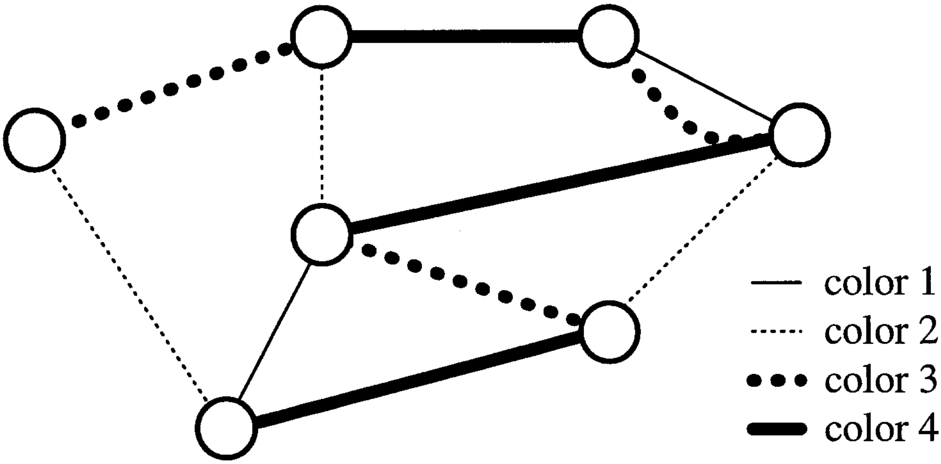

Fig. 1. An edge-coloring of a graph with four colors.

According to the rapid progress of computers, the research on computer algorithms has become active with emphasis on the efficiency and complexity, and efficient algorithms have been developed for various graph problems. However, Holyer [18] proved that the edge-coloring problem is NP-complete, and hence it is very unlikely that there is a polynomial time algorithm for solving the problem [1]. Hence a good approximation algorithm would be useful. Approximation algorithms are evaluated by the approximation ratio and the complexity. The polynomial time algorithm having the best approximate ratio so far was given by Nishizeki and Kashiwagi [27], whose approximation ratio is asymptotically 1.1. Gabow *et al.* [12] gave the most efficient algorithm which edge-colors a simple graph G with at most $\Delta(G) + 1$ colors in $O(m\sqrt{n \log n})$ time. Furthermore sequential and parallel algorithms have been obtained for various classes of graphs, such as bipartite graphs [8,11], planar graphs [6,7,12], series-parallel graphs [29,31,38,42,43], partial k-trees [4,36,37,40], degenerated graphs and bounded-genus graphs [39].

On the other hand, various generalizations of edge-coloring have been introduced and investigated. In 1970's Hilton and de Werra obtained many notable results on "equitable and edge-balanced colorings" in which each color appears at each vertex uniformly [16,34,35]. In 1980's Hakimi and Kariv studied the following f-coloring problem. An f-coloring of a graph G is a coloring of edges of

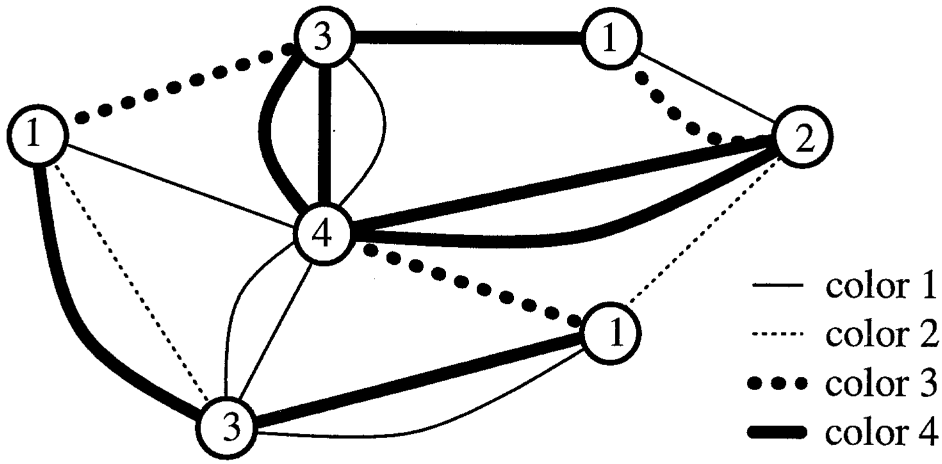

Fig. 2. An f-coloring of a graph with four colors.

G such that each color appears at each vertex $v \in V$ at most $f(v)$ times [15]. Fig. 2 depicts an f-coloring with four colors where numbers in circles mean $f(v)$ for vertices v. An ordinary edge-coloring is a special case of an f-coloring in which $f(v) = 1$ for every vertex $v \in V$. The minimum number of colors needed to f-color G is called the f-chromatic index $\chi'_f(G)$ of G. Since deciding the chromatic index of G is NP-complete [18], deciding the f-chromatic index of G is also NP-complete in general. On the other hand, various upper bounds on $\chi'_f(G)$ have been known [15,23,39].

The file transfer problem on computer networks introduced by Coffman *et al.*[5] is related to the edge-coloring problem. The file transfer problem is modeled as follows. Each computer v has a limited number $f(v)$ of communication ports. For each pair of computers there are a number of files which are transferred between the pair of computers. In such a situation the problem is how to schedule the file transfers so as to minimize the total time for the overall transfer process. The general problem is NP-complete since the simple version of the problem can be reduced to the edge-coloring problem. Coffman *et al.* obtained simple approximate algorithms for the file transfer problem. The file transfer problem in which each file has the same length is formulated as an f-coloring problem for a graph as follows. Vertices of the graph correspond to nodes of the network, and edges correspond to files to be transferred between the endpoints. Such a graph G describes the file transfer demands. Assume that each computer v has $f(v)$ communication ports, and transferring any file takes an equal amount of time. Under these assumptions, the schedule to minimize the total time for the overall transfer process corresponds to an f-coloring of G with the minimum number of colors. Note that the edges colored with the same color correspond to files that can be transferred simultaneously.

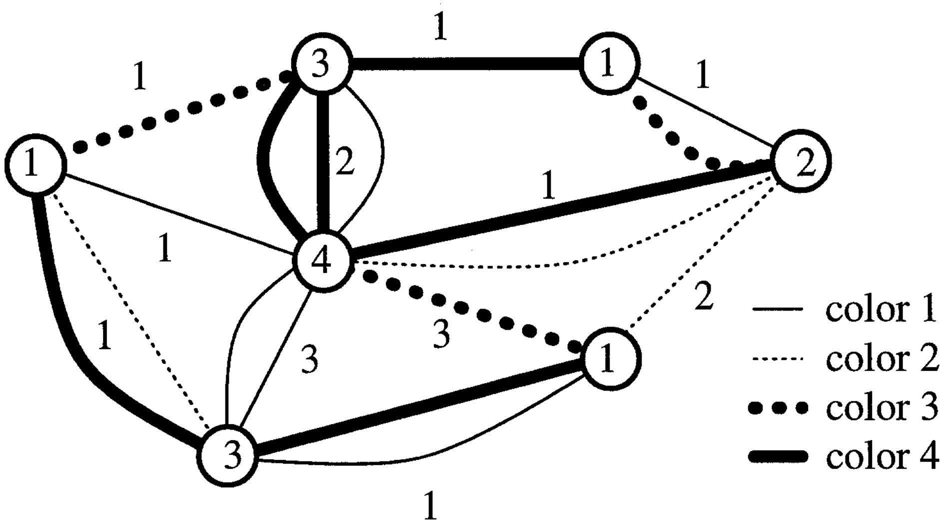

Fig. 3. An fg-coloring of a graph with four colors.

For some cases of the file transfer problem on computer networks, we must often consider the capacity of channels. An f-coloring is generalized to an fg-coloring so as to treat such a file transfer problem with channel constraints [24]. The capacity of a channel between vertices v and w is denoted by $g(vw)$. An fg-coloring of G is a coloring of edges such that each vertex v has at most $f(v)$ edges colored with the same color and each set $E(vw)$ of multiple edges contains at most $g(vw)$ edges colored with the same color. Fig. 3 depicts an fg-coloring with four colors, where numbers next to multiple edges $E(vw)$ mean $g(vw)$. The minimum number of colors needed to fg-color G is called the *fg-chromatic index* $\chi'_{fg}(G)$ of G. Several upper bounds on χ'_{fg} have been known [24,25].

1.2 Definitions

In this paper we deal with so-called *multigraphs* which may have multiple edges but have no selfloops. A graph in which at most one edge joins any pair of vertices is called a *simple* graph. $G = (V, E)$ denotes a graph with vertex set V and edge set E. We denote the degree of vertex v by $d(G, v)$ or simply by $d(v)$, and the *maximum degree* of G by $\Delta(G) = \max_{v \in V} d(v)$. An edge joining vertices v and w is denoted by vw. $E(vw)$ is the set of multiple edges joining vertices v

and w, and $p(vw)$ is the cardinality of the set $E(vw)$, that is $p(vw) = |E(vw)|$. We write $H \subseteq G$ if H is a subgraph of G. $V(H)$ denotes the set of vertices of H, and $E(H)$ denotes the set of edges of H. We use $\lfloor x \rfloor$ for the largest integer not greater than x, and $\lceil x \rceil$ for the smallest integer not smaller than x.

A *bipartite graph* $G = (V_1, V_2, E)$ is a graph whose vertex set can be partitioned into two sets V_1 and V_2 so that $e \in V_1 \times V_2$ for each edge $e \in E$.

The class of *k-trees* is defined recursively as follows:

(a) A complete graph with k vertices is a k-tree.

(b) If $G = (V, E)$ is a k-tree and k vertices $v_1, v_2, \cdots v_k$ induce a complete subgraph of G, then $G' = (V \cup \{w\}, E \cup \{wv_i | 1 \leq i \leq k\})$ is a k-tree where w is a new vertex not contained in G.

(c) All k-trees can be formed with rules (a) and (b).

A graph is a *partial k-tree* if it is a subgraph of a k-tree. Thus partial k-trees are simple graphs.

A *planar graph* is a simple graph which can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident.

The *genus* $g(G)$ of a simple graph G is the minimum number of handles which must be added to a sphere so that G can be embedded on the resulting surface. Of cource, $g(G) = 0$ if and only if G is planar.

Let s be a positive integer. A simple graph G is *s-degenerate* if the vertices of G can be ordered $v_1, v_2, \cdots, v_n$ so that $d(v_i, G_i) \leq s$ for each i, $1 \leq i \leq n$, where $G_0 = G$ and $G_i = G - \{v_1, v_2, \cdots, v_{i-1}\}$. That is, G is s-degenerate if and only if G can be reduced to the trivial (or degenerate) graph K_1 by the successive removal of vertices having degree at most s. Obviously a partial k-tree is k-degenerate, and a planar graph is 5-degenerate.

The *arboricity* of a simple graph G, denoted by $a(G)$, is the minimum number of edge-disjoint forests into which G can be decomposed.

The *thickness* of a simple graph G, denoted by $\theta(G)$, is the minimum number of planar subgraphs whose union is G.

2 Edge-coloring

2.1 Edge-colorings of multigraphs

Every bipartite multigraph G can be edge-colored with $\Delta(G)$ colors, that is $\chi'(G) \leq \Delta(G)$ [20]. On the other hand, clearly $\chi'(G) \geq \Delta(G)$. Therefore $\chi'(G) = \Delta(G)$. One can easily prove the inequality $\chi'(G) \leq \Delta(G)$ above using a technique of "switching an alternating path." The proof immediately yields an algorithm to edge-color a bipartite graph G with $\Delta(G)$ colors in time $O(mn)$. There exists a more efficient algorithm which, based on the divide and conquer, edge-colors a bipartite multigraph in time $O(m \log m)$ [8].

Vizing [32] obtained the following upper bound on the chromatic index of multigraphs:

$$\chi'(G) \leq \max_{v,w \in V} \{d(w) + p(vw)\}.$$

On the other hand, Shannon [30] proved that the following upper bound holds:

$$\chi'(G) \leq \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor.$$

Shannon's proof immediately yields an algorithm to edge-color a multigraph G with $\lfloor \frac{3}{2}\Delta(G) \rfloor$ colors in time $O(m(\Delta(G)+n))$.

There is another upper bound on the chromatic index of multigraphs [2]:

$$\chi'(G) \leq \max\left\{ l(G), \left\lfloor \frac{5\Delta(G)+2}{4} \right\rfloor \right\},$$

where

$$l(G) = \max_{H \subseteq G, |V(H)| \geq 3} \left\lceil \frac{|E(H)|}{\left\lfloor \frac{|V(H)|}{2} \right\rfloor} \right\rceil.$$

Clearly at most $\left\lfloor \frac{|V(H)|}{2} \right\rfloor$ edges of $|E(H)|$ edges in subgraph H can be colored with the same color. Therefore $l(G)$ is a lower bound on $\chi'(G)$: $\chi'(G) \geq l(G)$.

Furthermore the following better upper bounds are known [14, 17, 27]:

$$\chi'(G) \leq \max\left\{ l(G), \left\lfloor \frac{7\Delta(G)+4}{6} \right\rfloor \right\},$$
$$\chi'(G) \leq \max\left\{ l(G), \left\lfloor \frac{9\Delta(G)+6}{8} \right\rfloor \right\} \quad \text{and}$$
$$\chi'(G) \leq \max\left\{ l(G), \left\lfloor \frac{11\Delta(G)+8}{10} \right\rfloor \right\}.$$

An algorithm using colors no more than Vizing's upper bound has the approximation ratio 2 since

$$\max_{v,w \in V} \{d(w) + p(vw)\} \leq 2\chi'(G).$$

The algorithm using colors no more than Shannon's upper bound has the approximation ratio 3/2 since

$$\left\lfloor \frac{3}{2}\Delta(G) \right\rfloor \leq \frac{3}{2}\chi'(G).$$

Furthermore Nishizeki and Kashiwagi's algorithm using colors no more than $\max\{l(G), \lfloor (11\Delta+8)/10 \rfloor\}$ has the asymptotic approximation ratio 1.1 since

$$\max\left\{ l(G), \left\lfloor \frac{11\Delta(G)+8}{10} \right\rfloor \right\} \leq \frac{11\chi'(G)+8}{10}.$$

Table 1. Sequential algorithms.

Classes of graphs	Time	Colors	Refs.
Simple graph	$O(m\sqrt{n \log n})$	$\Delta + 1$	[12]
Multigraph	$O(m(\Delta + n))$	$1.1\chi' + 0.8$	[27]
Bipartite multigraph	$O(m \log m)$	Δ	[8]
Series-parallel multigraph	$O(m \log m)$	χ'	[42]
Partial k-tree	$O(n)$	χ'	[36]
Planar graph ($\Delta \geq 9$)	$O(n \log n)$	Δ	[6]
Planar graph ($\Delta \geq 19$)	$O(n)$	Δ	[7]
Genus $g \geq 1$ ($\Delta \geq 2\lfloor(5 + \sqrt{48g+1})/2\rfloor$	$O(n^2)$	Δ	[39]
Genus $g \geq 1$ ($\Delta \geq \lceil(9 + \sqrt{48g+1})^2/8\rceil$	$O(n \log n)$	Δ	[39]
Degeneracy s ($\Delta \geq 2s$)	$O(n^2)$	Δ	[39]
Degeneracy s ($\Delta \geq \lceil(s+2)^2/2\rceil - 1$)	$O(n \log n)$	Δ	[39]
Arboricity a ($\Delta \geq 4a - 2$)	$O(n^2)$	Δ	[39]
Arboricity a ($\Delta \geq \lceil(a+2)^2/2\rceil - 1$)	$O(n \log n)$	Δ	[39]
Thickness θ ($\Delta \geq 12\theta - 2$)	$O(n^2)$	Δ	[39]
Thickness θ ($\Delta \geq \lceil(3\theta+2)^2/2\rceil - 1$)	$O(n \log n)$	Δ	[39]

Table 2. *NC* parallel algorithms.

Classes of graphs	Parallel time	Operations	Colors	Refs.
Bipartite multigraph	$O(\log^3 n)$	$O(m)$	Δ	[21]
Series-parallel multigraph	$O(\log n)$	$O(n\Delta)$	χ'	[43]
Partial k-tree	$O(\log n)$	$O(n)$	χ'	[37]
Planar graph ($\Delta \geq 9$)	$O(\log^3 n)$	$O(n \log^3 n)$	Δ	[6]
Planar graph ($\Delta \geq 19$)	$O(\log^2 n)$	$O(n \log^2 n)$	Δ	[7]
Genus $g \geq 1$ $\Delta \geq \lceil(9 + \sqrt{48g+1})^2/8\rceil$	$O(\log^3 n)$	$O(n \log^3 n)$	Δ	[39]
Degeneracy s ($\Delta \geq \lceil(s+2)^2/2\rceil - 1$)	$O(\log^3 n)$	$O(n \log^3 n)$	Δ	[39]
Arboricity a ($\Delta \geq \lceil(a+2)^2/2\rceil - 1$)	$O(\log^3 n)$	$O(n \log^3 n)$	Δ	[39]
Thickness θ ($\Delta \geq \lceil(3\theta+2)^2/2\rceil - 1$)	$O(\log^3 n)$	$O(n \log^3 n)$	Δ	[39]

On the other hand, Goldberg [14] posed a conjecture that for every odd integer $k \geq 5$

$$\chi'(G) \leq \max\left\{l(G), \left\lfloor \frac{k\Delta(G) + (k-3)}{k-1} \right\rfloor\right\}.$$

Furthermore Goldberg [14] and Seymour [28] posed a conjecture that every multigraph G satisfies

$$\chi'(G) \leq \max\{l(G), \Delta(G) + 1\}.$$

Especially for series-parallel multigraph G it is known [22,29] that

$$\chi'(G) = \max\{l(G), \Delta(G)\},$$

and that there exists an algorithm to decide $\chi'(G)$ in linear time and find an edge-coloring of G with $\chi'(G)$ colors in time $O(m \log m)$ [42,43].

2.2 Edge-colorings of simple graphs

Vizing [32] proved that every simple graph G can be edge-colored with $\Delta(G)+1$ colors, that is,

$$\chi'(G) \leq \Delta(G) + 1.$$

Since $\chi'(G) \geq \Delta(G)$, every simple graph G can be edge-colored with either $\Delta(G)$ or $\Delta(G) + 1$ colors. His proof depends on a technique of "shifting a fan." The proof immediately yields an $O(mn)$ time algorithm to edge-color a simple graph G with $\Delta(G) + 1$ colors. There exist more efficient edge-coloring algorithms for simple graphs. The most efficient one takes time $O(m\sqrt{n \log n})$ [12].

The following results are known for planar graphs G. If $2 \leq \Delta(G) \leq 5$ then $\chi'(G) = \Delta(G)$ or $\Delta(G)+1$. If $\Delta(G) \geq 8$ then $\chi'(G) = \Delta(G)$. It is open whether there exists a planar graph G with $6 \leq \Delta(G) \leq 7$ such that $\chi'(G) = \Delta(G) + 1$. For the case $\Delta(G) \geq 8$ there is an $O(n^2)$ time algorithm for edge-coloring a planar graph G with $\Delta(G)$ colors [31]. Furthermore for the case $\Delta(G) \geq 9$ there is a more efficient algorithm of time-complexity $O(n \log n)$ [6].

There are efficient algorithms to edge-color various classes of graphs G with $\chi'(G)$ colors if $\Delta(G)$ is large. These sequential algorithms together with others are listed in Table 1, and NC parallel algorithms in Table 2. It has not been known whether there is an NC parallel algorithm to edge-color any simple graph G with $\Delta(G) + 1$ colors [19].

3 f-coloring

The *vertex-capacity* f is any function from the vertex set V to the natural numbers. An *f-coloring* of a graph G is a coloring of the edges of G such that each vertex v has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is called the *f-chromatic index* of G, and is denoted by $\chi'_f(G)$. Fig. 2 depicts an f-coloring with four colors, and $\chi'_f(G) = 4$ since there exists a vertex v such that $f(v) = 1$ and $d(v) = 4$. Hakimi and Kariv obtained the following upper bound on $\chi'_f(G)$:

$$\chi'_f(G) \leq \max_{v,w \in V} \left\lceil \frac{d(v) + p(vw)}{f(v)} \right\rceil.$$

This upper bound is a generalization of Vizing's bound for the ordinary edge-coloring because an ordinary edge-coloring is a special case of an f-coloring in which $f(v) = 1$ for every vertex $v \in V$. Their proof uses an extended version of switching an alternating path and shifting a fan. The proof immediately yields an $O(m^2)$ time algorithm to find an f-coloring of G with a number of colors not exceeding the upper bound. Furthermore the time complexity was improved to $O(m\sqrt{m\log m})$ [26]. The following upper bound on $\chi'_f(G)$ is also known [23]:

$$\chi'_f(G) \leq \max\left\{l_f(G), \left\lfloor \frac{9\Delta_f(G)+6}{8} \right\rfloor\right\}$$

where

$$l_f(G) = \max_{H\subseteq G, |V(H)|\geq 3} \left\lceil \frac{|E(H)|}{\left\lfloor \frac{\sum_{v\in V(H)} f(v)}{2} \right\rfloor} \right\rceil$$

and

$$\Delta_f(G) = \max_{v\in V} \left\lceil \frac{d(v)}{f(v)} \right\rceil.$$

Note that $l_f(G)$ and $\Delta_f(G)$ are trivial lower bounds on $\chi'_f(G)$. Furthermore an algorithm is known to find an f-coloring of G with a number of colors not exceeding the upper bound above in time $O(m^2)$ [23]. This algorithm has the asymptotic approximation ratio 9/8.

Since both the ordinary edge-coloring problem and the f-coloring problem are NP-complete, the theory of NP-completeness immediately implies that there exists a polynomial-time reduction of the f-coloring problem to the ordinary edge-coloring problem plausibly through SAT. Recently a very simple reduction of the f-coloring problem to the ordinary edge-coloring problem was found [41].

4 *fg*-coloring

The *edge-capacity* g is any function from the pairs of vertices $V\times V$ to the natural numbers. An *fg-coloring* of G is a coloring of the edges of G such that at most $f(v)$ edges incident to v are colored with the same color for each vertex $v \in V$ and at most $g(vw)$ multiple edges joining v and w are colored with the same color for each pair of vertices v and w. The minimum number of colors needed to fg-color G is called the *fg-chromatic index* of G, and is denoted by $\chi'_{fg}(G)$. The fg-chromatic index of G in Fig. 3 is four. The following upper bound on $\chi'_{fg}(G)$ has been known [25]:

$$\chi'_{fg}(G) \leq \left\lfloor \frac{3}{2}\Delta_{fg}(G) \right\rfloor,$$

where

$$\Delta_{fg}(G) = \max\left\{\Delta_f(G), \Delta_g(G)\right\},$$

$$\Delta_f(G) = \max_{v\in V} \left\lceil \frac{d(v)}{f(v)} \right\rceil,$$

and

$$\Delta_g(G) = \max_{vw\in E} \left\lceil \frac{p(vw)}{g(vw)} \right\rceil.$$

Note that $\Delta_f(G)$, $\Delta_g(G)$ and $\Delta_{fg}(G)$ are trivial lower bounds on $\chi'_{fg}(G)$. This upper bound is a generalization of Shannon's bound [30] for the ordinary edge-coloring. An $O(m^2)$ time algorithm to fg-color G with a number of colors not exceeding $\lfloor \frac{3}{2}\Delta_{fg}(G) \rfloor$ has been known [25]. Since

$$\left\lfloor \frac{3}{2}\Delta_{fg}(G) \right\rfloor \le \frac{3}{2}\chi'_{fg}(G),$$

this algorithm has the approximation ratio 3/2.

One may assume without loss of generality that $g(vw) \le \max\{f(v), f(w)\}$ for any pair of vertices $v, w \in V$. Then the following upper bound holds for the fg-chromatic index $\chi'_{fg}(G)$ [24]:

$$\chi'_{fg}(G) \le \max_{v,w\in V} \left\lceil \frac{d(v)}{f(v)} + \frac{p(vw)}{g(vw)} \right\rceil.$$

Although one may assume that $g(vw) \le \min\{f(v), f(w)\}$, we assume as above since the bound would increase when g decreases. This upper bound is a generalization of Vizing's upper bound [32] for the ordinary edge-coloring and Hakimi and Kariv's [15] upper bound for the f-coloring. The proof for this upper bound is constructive, and immediately yields an $O(m^2)$ time algorithm to fg-color a given graph with a number of colors not exceeding the upper bound above. Since

$$\max_{v,w\in V} \left\lceil \frac{d(v)}{f(v)} + \frac{p(vw)}{g(vw)} \right\rceil \le 2\chi'_{fg}(G),$$

this algorithm has the approximation ratio 2.

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