Canonical Decomposition, Realizer, Schnyder Labeling and Orderly Spanning Trees of Plane Graphs
(Extended Abstract)

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Abstract. A canonical decomposition, a realizer, a Schnyder labeling and an orderly spanning tree of a plane graph play an important role in straight-line grid drawings, convex grid drawings, floor-plannings, graph encoding, etc. It is known that the triconnectivity is a sufficient condition for their existence, but no necessary and sufficient condition has been known. In this paper, we present a necessary and sufficient condition for their existence, and show that a canonical decomposition, a realizer, a Schnyder labeling, an orderly spanning tree, and an outer triangular convex grid drawing are notions equivalent with each other. We also show that they can be found in linear time whenever a plane graph satisfies the condition.

1 Introduction

Recently automatic aesthetic drawing of graphs has created intense interest due to their broad applications, and as a consequence, a number of drawing methods have come out [3–9, 11–13, 15, 18, 21, 23]. The most typical drawing of a plane graph G is the *straight line drawing* in which all vertices of G are drawn as points and all edges are drawn as straight line segments without any edge-intersection. A straight line drawing of G is called a *grid drawing* if the vertices of G are put on grid points of integer coordinates. A straight line drawing of G is called a *convex drawing* if every face boundary is drawn as a convex polygon [4, 22, 23]. A convex drawing of G is called an *outer triangular convex drawing* if the outer face boundary is drawn as a triangle, as illustrated in Fig. 1.

A canonical decomposition, a realizer, a Schnyder labeling and an orderly spanning tree of a plane graph G play an important role in straight-line drawings, convex grid drawings, floor-plannings, graph encoding, etc. [1, 2, 5–8, 12, 13, 15–18, 21]. It is known that the triconnectivity is a sufficient condition for their existence in G [5, 10, 12, 21], but no necessary and sufficient condition has been known. In this paper, we present a necessary and sufficient condition for their existence, and show that a canonical decomposition, a realizer, a Schnyder

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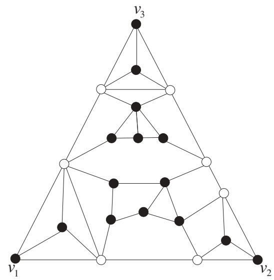

Fig. 1. An outer triangular convex drawing of a plane graph.

labeling, an orderly spanning tree, and an outer triangular convex grid drawing are notions equivalent with each other. We also show that they can be found in linear time whenever G satisfies the condition. Algorithms for finding them have only been available for the triconnected case.

2 Main Theorem and Definitions

Let G be a plane biconnected simple graph. We assume for simplicity that the degrees of all vertices are larger than or equal to three, since the two edges adjacent to a vertex of degree two are often drawn on a straight line. Our main result is the following theorem; some terms will be defined later.

Theorem 1. Let v_1, v_2 and v_3 be vertices on the outer face $F_o(G)$ of G appearing counterclockwise in this order, as illustrated in Fig. 2. Let P_1 be the path from v_1 to v_2 on $F_o(G)$, let P_2 be the path from v_2 to v_3, and let P_3 be the path from v_3 to v_1. Then the following six propositions (a)–(f) are equivalent with each other.

(a) G has a canonical decomposition with respect to v_1, v_2 and v_3.
(b) G has a realizer with respect to v_1, v_2 and v_3.
(c) G has a Schnyder labeling with respect to v_1, v_2 and v_3.
(d) G has an outer triangular convex grid drawing such that $F_o(G)$ is drawn as a triangle $v_1v_2v_3$.
(e) G is internally triconnected, and has no separation pair $\{u, v\}$ such that both u and v are on the same path P_i, $1 \leq i \leq 3$. (See Figs. 1,2,3,4.)
(f) G has an orderly spanning tree such that v_3 is the root, v_1 is the minimum leaf, and v_2 is the maximum leaf.

It is known that (a)$\Rightarrow$(b), (b)$\Leftrightarrow$(c) and (c)$\Rightarrow$(d) [10, 12, 21]. Furthermore, (d)$\Rightarrow$(e) holds as shown later in Lemma 1. In this paper, we complete a proof of Theorem 1 by proving (e)$\Rightarrow$(a) in Section 3 and (b)$\Leftrightarrow$(f) in Section 4.

In the remainder of this section, we present some definitions and known lemmas.

We denote by $G = (V, E)$ an undirected simple graph with vertex set V and edge set E. Let n be the number of vertices of G. An undirected edge joining vertices u and v is denoted by (u, v). We denote by $\langle u, v\rangle$ a directed edge from u to v.

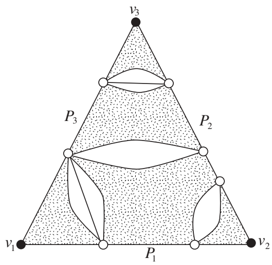

Fig. 2. Illustration for the necessary and sufficient condition.

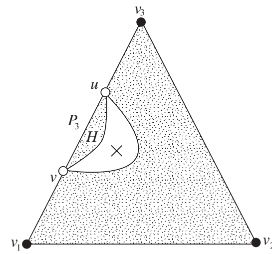

Fig. 3. A separation pair $\{u, v\}$ on P_3.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane graph* is a planar graph with a fixed embedding. A plane graph divides the plane into connected regions called *faces*. We denote by $F_o(G)$ the outer face of G. The boundary of $F_o(G)$ is also denoted by $F_o(G)$. A vertex on $F_o(G)$ is called an *outer vertex*, while a vertex not on $F_o(G)$ is called an *inner vertex*. An edge on $F_o(G)$ is called an *outer edge*, while an edge not on $F_o(G)$ is called an *inner edge*.

We call a vertex v of G a *cut vertex* if its removal from G results in a disconnected graph. A graph G is *biconnected* if G has no cut vertex. We call a pair $\{u, v\}$ of vertices in a biconnected graph G a *separation pair* if its removal from G results in a disconnected graph, that is, $G - \{u, v\}$ is not connected. A biconnected graph G is *triconnected* if G has no separation pair. A plane biconnected graph G is *internally triconnected* if, for any separation pair $\{u, v\}$ of G, both u and v are outer vertices and each connected component of $G - \{u, v\}$ contains an outer vertex. In other words, G is internally triconnected if and only if it can be extended to a triconnected graph by adding a vertex in an outer face and joining it to all outer vertices. An internally triconnected plane graph is depicted in Fig. 1, where all the vertices of separation pairs are drawn as white circles. If a biconnected plane graph G is not internally triconnected, then G has a separation pair $\{u, v\}$ as illustrated in Figs. 4(a)–(c) and a "split component" H contains a vertex other than u and v.

We now have the following lemma.

Lemma 1. (d)$\Rightarrow$(e).

Proof. Suppose that a biconnected plane graph G is not internally triconnected. Then G has a separation pair $\{u, v\}$ as illustrated in Figs. 4(a)–(c), and a "split component" H has a vertex w other than u and v. The degree of w is larger than or equals to three by the assumption of this paper, and hence the two faces marked by $\times$ cannot be simultaneously drawn as convex polygons. Thus G has no outer triangular convex drawing.

Suppose that G has a separation pair $\{u, v\}$ such that both u and v are on P_i, $1 \leq i \leq 3$, as illustrated in Fig. 3. Then G has no outer triangular

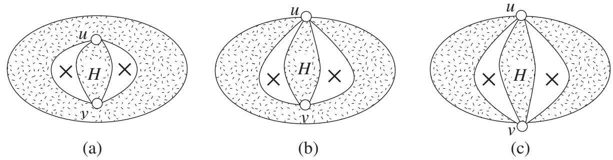

Fig. 4. Biconnected plane graphs which are not internally triconnected.

convex drawing, because P_i must be drawn as a straight line segment, the split component H has a vertex of degree three or more other than u and v, and hence the face marked by $\times$ in Fig. 3 cannot be drawn as a convex polygon.

The two facts above immediately implies (d)$\Rightarrow$(e). $\square$

We then define a canonical decomposition [5]. Let $V = \{u_1, u_2, \cdots, u_n\}$. Let u_1, u_2 and u_n be three outer vertices appearing counterclockwise on $F_o(G)$ in this order. We may assume that u_1 and u_2 are consecutive on $F_o(G)$; otherwise, let G be the graph obtained by adding a virtual edge (u_1, u_2) to the original graph. Let $\Pi = (V_1, V_2, \cdots, V_h)$ be an ordered partition of V into nonempty subsets $V_1, V_2, \cdots, V_h$. Then $V_1 \bigcup V_2 \bigcup \cdots \bigcup V_h = V$ and $V_i \bigcap V_j = \emptyset$ for any indices i and j, $1 \le i < j \le h$. We denote by G_k, $1 \le k \le h$, the subgraph of G induced by $V_1 \bigcup V_2 \bigcup \cdots \bigcup V_k$, and by $\overline{G_k}$ the subgraph of G induced by $V_{k+1} \bigcup V_{k+2} \bigcup \cdots \bigcup V_h$. Note that $G = G_h$. We say that Π is a *canonical decomposition* of G (with respect to u_1, u_2 and u_n) if the following three conditions (cd1)–(cd3) hold:

(cd1) V_1 consists of all the vertices on the boundary of the inner face containing the outer edge (u_1, u_2), and $V_h = \{u_n\}$.
(cd2) For each index k, $1 \le k \le h$, G_k is internally triconnected.
(cd3) For each index k, $2 \le k \le h$, all the vertices in V_k are outer vertices of G_k, and
  (a) if $|V_k| = 1$, then the vertex w in V_k has two or more neighbors in G_{k-1} and has at least one neighbor in $\overline{G_k}$ when $k < h$; and
  (b) if $|V_k| \ge 2$, then the vertices in V_k consecutively appear on $F_o(G_k)$, each of the first and last vertices in V_k has exactly one neighbor in G_{k-1}, and all the other intermediate vertices in V_k have no neighbor in G_{k-1}, and each vertex in V_k has at least one neighbor in $\overline{G_k}$.

A canonical decomposition of the graph in Fig. 1 is illustrated in Fig. 5. Although the definition of a canonical decomposition above is slightly different from one in [5], they are effectively equivalent with each other. The following lemma is known.

Lemma 2. [5] Every triconnected plane graph G has a canonical decomposition.

The definition of a realizer [10] is omitted in this extended abstract, due to the page limitation. The following lemma is known on a realizer and a canonical decomposition.

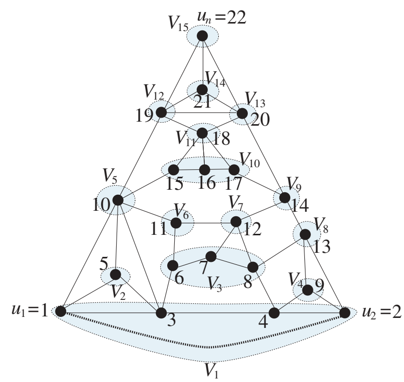

Fig. 5. A canonical decomposition $\Pi = (V_1, V_2, \cdots, V_{15})$ of the graph G in Fig. 1.

Lemma 3. [10] If a plane graph G has a canonical decomposition with respect to u_1, u_2 and u_n, then G has a realizer with respect to u_1, u_2 and u_n.

The definition of a Schnyder labeling [12, 21] is omitted in this extended abstract, due to the page limitation. The following three lemmas are known on the Schnyder labeling.

Lemma 4. [12, 21] Every triconnected plane graph G has a Schnyder labeling, and it can be computed in linear time.

Lemma 5. [12] A plane graph G has a realizer if and only if G has a Schnyder labeling.

Lemma 6. [12] If a plane graph G has a Schnyder labeling with respect to a_1, a_2 and a_3, then G has an outer triangular convex grid drawing such that $F_o(G)$ is drawn as a triangle $a_1a_2a_3$ and the size of grid is $(n-1) \times (n-1)$.

We then define an orderly spanning tree [1, 2, 17]. Let T be a spanning tree of G rooted at an outer vertex u_1 of G. Let $u_1, u_2, \cdots, u_n$ be the counterclockwise preordering of vertices in T as illustrated in Fig. 6, where T is drawn by thick lines and each vertex u_i, $1 \leq i \leq n$, is attached an index i. We call the leaf u_α of T having the maximum index the *maximum leaf* of T. Clearly $\alpha = n$. We call the leaf u_β having the minimum index the *minimum leaf* of T. We say that two distinct vertices of G are *unrelated* if any of them is not an ancestor of the other in T. Let $N(u_i)$ be the set of all the neighbors of u_i in G. The set $N(u_i)$ is partitioned into the following four subsets $N_1(u_i), N_2(u_i), N_3(u_i)$ and $N_4(u_i)$:

$$
\begin{aligned}
N_1(u_i) &= \{u_j \in N(u_i) \mid u_j \text{ is the parent of } u_i \text{ in } T\},\\
N_2(u_i) &= \{u_j \in N(u_i) \mid j < i,\ u_j \text{ is not the parent of } u_i\},\\
N_3(u_i) &= \{u_j \in N(u_i) \mid u_j \text{ is a child of } u_i \text{ in } T\}, \text{ and}\\
N_4(u_i) &= \{u_j \in N(u_i) \mid j > i,\ u_j \text{ is not a child of } u_i \text{ in } T\}.
\end{aligned}
$$

Note that $N_1(u_1) = \emptyset$, $|N_1(u_i)| = 1$ for each vertex u_i, $2 \leq i \leq n$, and $|N_3(u_i)| = \emptyset$ for each leaf u_i of T. We call T an *orderly spanning tree* of G if the following conditions (ost1) and (ost2) hold:

(ost1) For each vertex u_i, $1 \leq i \leq n$, u_i and any vertex $u_j \in N_2(u_i) \bigcup N_4(u_i)$ are unrelated, and the vertices in $N_1(u_i), N_2(u_i), N_3(u_i)$ and $N_4(u_i)$ appear around u_i counterclockwise in this order, as illustrated in Fig. 7; and
(ost2) For each leaf u_i of T other than u_α and u_β, $N_2(u_i), N_4(u_i) \neq \emptyset$.

Chiang *et al.* [2] define an orderly spanning tree only for maximal plane graphs, and there is no Condition (ost2) in their definition. They show that a maximal plane graph G has an orderly spanning tree if and only if G has a realizer [2]. We add Condition (ost2) since G is not necessarily a maximal plane graph in this paper.

We have (a)⇒(b) by Lemma 3, (b)⇔(c) by Lemma 5, (c)⇒(d) by Lemma 6, and (d)⇒(e) by Lemma 1. In order to prove Theorem 1, it suffices to prove (e)⇒(a) and (b)⇔(f).

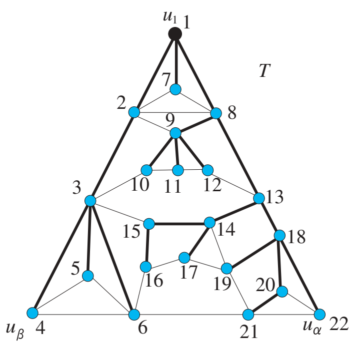

Fig. 6. An orderly spanning tree of the graph G in Fig. 1.

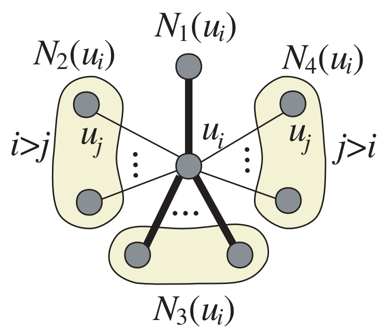

Fig. 7. Four sets $N_1(u_i)$, $N_2(u_i)$, $N_3(u_i)$ and $N_4(u_i)$.

3 Proof of (e)⇒(a)

In this section, we give a proof of (e)⇒(a). We first define some terms.

If an internally triconnected plane graph G is not triconnected, then G has a separation pair of outer vertices and hence has a "chord path" defined below when G is not a single cycle.

Let G be a biconnected plane graph, and let $w_1, w_2, \cdots, w_t$ be the vertices appearing on $F_o(G)$ clockwise in this order. We call a path Q in G a *chord-path* if Q satisfies the following (i)–(iv):

(i) Q connects two outer vertices w_p and w_q, $p < q$;
(ii) $\{w_p, w_q\}$ is a separation pair of G;
(iii) Q lies on an inner face; and

(iv) Q does not pass through any outer edge and any outer vertex other than the ends w_p and w_q.

A chord-path Q connecting w_p and w_q is *minimal* if none of $w_{p+1}, w_{p+2}, \cdots, w_{q-1}$ is an end of a chord-path. Thus the definition of a minimal chord-path depends on which vertex is considered as the starting vertex w_1 of $F_o(G)$.

Let $\{x_1, x_2, \cdots, x_p\}$, $p \geq 3$, be a set of three or more outer vertices consecutive on $F_o(G)$ such that $d(x_1) \geq 3$, $d(x_2) = d(x_3) = \cdots = d(x_{p-1}) = 2$, and $d(x_p) \geq 3$. Then we call the set $\{x_2, x_3, \cdots, x_{p-1}\}$ an *outer chain* of G.

We are now ready to prove (e)$\Rightarrow$(a).

[Proof of (e)$\Rightarrow$(a)]. Assume that the proposition (e) holds for G, that is, G is internally triconnected and has no separation pair $\{u, v\}$ such that both u and v are on the same path P_i, $1 \leq i \leq 3$, where P_1 connects v_1 and v_2, P_2 connects v_2 and v_3, and P_3 connects v_3 and v_1. We shall show that G has a canonical decomposition $\Pi = (V_1, V_2, \cdots, V_h)$. If the outer vertices v_1 and v_2 are not consecutive on $F_o(G)$, then add a virtual edge (v_1, v_2) to the original graph and let G be the resulting graph. Take $u_1 = v_1, u_2 = v_2$, and $u_n = v_3$. Take as V_1 the set of all the vertices on the boundary of the inner face containing the edge (u_1, u_2), and take $V_h = \{u_n\}$. (See Fig. 5.) Then $u_n = v_3 \notin V_1$; otherwise, $\{v_1, v_3\}$ would be a separation pair of G on P_3, a contradiction. Hence Condition (cd1) of a canonical decomposition holds. Since $G_h = G$ is internally triconnected, Condition (cd2) holds for $k = h$. Since $u_n = v_3$ has degree three or more, Condition (cd3) holds for $k = h$. G is internally triconnected, and the outer vertex u_n of G is not contained in any separation pair of G since (v_1, v_2) is an edge of G and $\{v_3, x\}$ is not a separation pair of G for any vertex x on path P_2 or P_3. Therefore $G_{h-1} = G - u_n$ is also internally triconnected, and hence (cd2) holds for $k = h - 1$. If $V = V_1 \bigcup V_h$, then simply setting $h = 2$ we can complete a proof. One may thus assume that $V \neq V_1 \bigcup V_h$ and hence $h \geq 3$. We choose $V_{h-1}, V_{h-2}, \cdots, V_2$ in this order and show that (cd2) and (cd3) hold.

Assume as an inductive hypothesis that $h \geq i + 1 \geq 3$ and the sets $V_h, V_{h-1}, \cdots, V_{i+1}$ have been appropriately chosen so that

(1) (cd2) holds for each index $k \geq i$, and
(2) (cd3) holds for each index $k \geq i + 1$.

We then show that there is a set V_i of outer vertices of G_i such that

(1) (cd2) holds for the index $k = i - 1$, and
(2) (cd3) holds for the index $k = i$.

Let $w_1, w_2, \cdots, w_t$ be the outer vertices of G_i appearing clockwise on $F_o(G)$ in this order, where $w_1 = v_1$ and $w_t = v_2$. There are the following two cases to consider.

Case 1: G_i is triconnected. Since G_i is triconnected and at least one vertex in V_{i+1} has a neighbor in G_i, there is an outer vertex $w \notin V_1$ of G_i which has a neighbor in $\overline{G_i}$. We choose the singleton set $\{w\}$ as V_i. Since G_i is triconnected and w is an outer vertex of G_i, $G_{i-1} = G_i - w$ is internally triconnected and w has three or more neighbors in G_{i-1}. Thus (cd2) holds for $k = i - 1$, and (cd3) holds for $k = i$.

Case 2: Otherwise. Since $i \geq 2$, G_i is not a single cycle. G_i is internally triconnected, but is not triconnected. Therefore there is a chord-path for $F_o(G_i)$. Let Q be a minimal chord-path, let w_p and w_q be the two ends of Q, and let $p < q$. Then $q \geq p + 2$; otherwise, G_i would not be internally triconnected. We now have the following two subcases.

Subcase 2a: $\{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ is an outer chain of G_i. In this case we choose $\{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ as V_i. Since V_i is an outer chain and Q is a minimal chord-path, we have $V_i \bigcap V_1 = \emptyset$. Each of w_{p+1} and w_{q-1} has exactly one neighbor in G_{i-1}, and each vertex $w \in V_i$ has a neighbor in $\overline{G_i}$ because w has degree three or more in G and has degree two in G_i. One can thus know that (cd3) holds for $k = i$.

We next show that (cd2) holds for $k = i - 1$. Assume for a contradiction that G_{i-1} is not internally triconnected. Then G_{i-1} has either a cut vertex v or a separation pair $\{u, v\}$ having one of the three types illustrated in Fig. 4.

Consider first the case where G_{i-1} has a cut vertex v, as illustrated in Fig. 8. Then v must be an outer vertex of G_i and $v \neq w_p, w_q$; otherwise, G_i would not be internally triconnected. The minimal chord-path Q above must pass through the outer vertex v, contrary to Condition (iv) of the definition of a chord-path.

Consider next the case where G_{i-1} has a separation pair $\{u, v\}$ having one of the three types illustrated in Fig. 4. Then $\{u, v\}$ would be a separation pair of G_i having one of the three types illustrated in Fig. 4, and hence G_i would not be internally triconnected, a contradiction.

Subcase 2b: Otherwise. In this case, every vertex in $\{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ has degree three or more in G_i; otherwise, Q would not be minimal. Furthermore, we can show that at least one vertex w in $\{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ has a neighbor in $\overline{G_i}$, as follows. Suppose for a contradiction that none of the vertices in $\{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ has a neighbor in $\overline{G_i}$. Then $\{w_p, w_q\}$ is a separation pair of G, and $w_p, w_{p+1}, \cdots, w_q$ is a path on $F_o(G)$, as illustrated in Fig. 9. Since $i \leq h - 1$, none of $w_p, w_{p+1}, \cdots, w_q$ is $v_3 = u_n$. Thus both w_p and w_q are either on path P_2 or on path P_3, and hence Proposition (e) would not hold for G, a contradiction.

We choose the singleton set $\{w\}$ as V_i for the vertex w above. Then clearly $V_i \bigcap V_1 = \emptyset$, and (cd3) holds for the index $k = i$. Since w is not an end of a chord-path of $F_o(G_i)$ and G_i is internally triconnected, $G_{i-1} = G_i - w$ is internally triconnected and hence (cd2) holds for the index $k = i - 1$. □

4 Proof of (b)⇔(f)

In this section, we give a proof of (b)⇔(f).

Sketchy proof of (b)⇔(f). We can prove that if (T_r, T_b, T_g) is a realizer of G then T_g is an orderly spanning tree of G rooted at r_g. The detail is omitted in this extended abstract, due to the page limitation.

From an orderly spanning tree T of G we can construct a realizer (T_r, T_b, T_g) of G. We take $T_g = T$. For each vertex u_i, $1 \leq i \leq n$, we appropriately choose

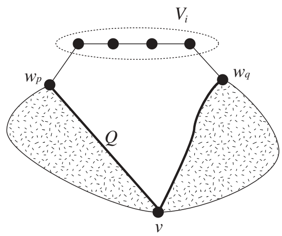

Fig. 8. Graph G_i and the outer chain V_i.

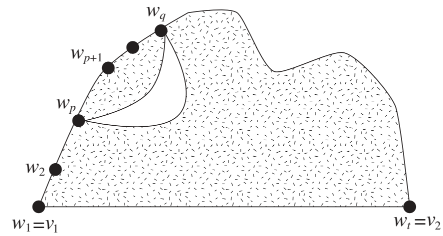

Fig. 9. Graph G_i.

an edge incident to u_i as an outgoing edge of T_b (or T_r) according to some rules. The detail is omitted in this extended abstract, due to the page limitation. $\Box$

Since the proof of (e)⇒(a) is constructive, we can immediately have a linear algorithm to find a canonical decomposition of G if Proposition (e) holds for G. Moreover, we can examine whether Proposition (e) holds for a given graph G, by using the linear algorithm for decomposing a graph to triconnected components [14]. Furthermore, one can know from the proof of (b)⇒(f) that if G has a realizer then one can immediately find an orderly spanning tree of G. We thus have the following corollary from Theorem 1.

Corollary 1. If a plane graph G is internally triconnected and has no separation pair $\{u, v\}$ such that both u and v are on the same path P_i, $1 \le i \le 3$, then one can find in linear time a canonical decomposition, a realizer, a Schnyder labeling, an orderly spanning tree, and an outer triangular convex grid drawing of G having size $(n-1) \times (n-1)$.

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