LP and NLP Optimization

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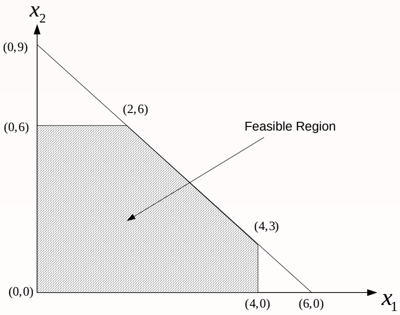
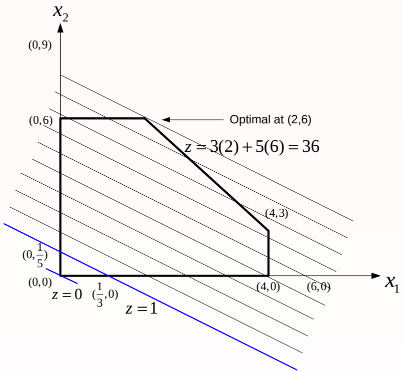
Linear Programming





- Max $Z = 3x_1 + 5x_2$

$$
\begin{aligned}
S.T: x_1 &\le 4, \\
2x_2 &\le 12, \\
3x_1 + 2x_2 &\le 18, \\
x_1 &\ge 0, \\
x_2 &\ge 0
\end{aligned}
$$

- Optimality Test for CPF(corner point feasible) solution
    - Consider any LP problem that possesses at least one optimal solution. If a CPF solution has no adjacent CPF solutions that are better, then it must be an optimal solution.
    - The simplex method focuses only on CPF solution (start from some simple CPF solution such as $(0, 0)$), find a better CPF solution until the best one is found (*i.e* moving from CPF solution to CPF solution).

Simplex Method

- Simplex method: Basically solving system of equations with slack variables

- Original Problem
  Max $Z = 3x_1 + 5x_2$

$$
\begin{aligned}
S.T: x_1 &\leq 4, \\
2x_2 &\leq 12, \\
3x_1 + 2x_2 &\leq 18, \\
x_1 &\geq 0, \\
x_2 &\geq 0,
\end{aligned}
$$

- Augmented Form
  Max
  $Z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5$

$$
\begin{aligned}
S.T: x_1 + x_3 &= 4, \\
2x_2 + x_4 &= 12, \\
3x_1 + 2x_2 + x_5 &= 18, \\
x_i &\geq 0 \text{ for all } i
\end{aligned}
$$

Simplex Method

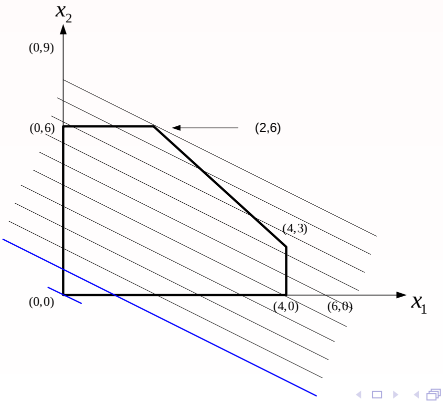
- Using the simplex method
  start from $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 4, 12, 18)$

  after several iterations,we get
  $x^* = (2, 6, 2, 0, 0)$
  $\therefore Z^* = 3(2) + 5(6) = 36.$
- The simplex method also produces shadow-prices which are useful for resource allocation by the manager.

Simplex Method

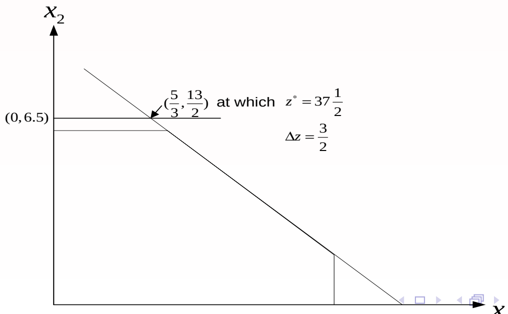
- In the example, $b_1 = 4, b_2 = 12, b_3 = 18$ (allocated resource).
- Shadow price for resource i (y_i^*) measures the rate at which Z could be increased by slightly increasing b_i (i.e., $\frac{dZ}{db_i}$).

In the example, $y_1^* = 0, y_2^* = 1.5, y_3^* = 1$



Simplex Method

- If $b_1 = 4 \to b_1 = 5$, no increase in $Z, i.e: y_1^* = 0$, since the 1^{st} constraint is not binding at x^*.
- If $b_2 = 12 \to b_2 = 13$, Z increases by 1.5 (when b_2 finally increases to $b_2 = 18, x^* = (0, 9)$ and further increase in b_2 does not increase Z).
- If $b_3 = 18 \to b_3 = 19$, $\triangle Z = 1$, because $y_3^* = 1$.



Simplex Method

The shadow prices are also produced from the simplex method

$$\begin{array}{cccccc} & & \text{Coefficients} & & & \\ Z & x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & -3 & -5 & 0 & 0 & 0 \end{array} \rightarrow \begin{array}{cccccc} & & \text{Coefficients} & & & \\ Z & x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & -3 & 0 & 0 & 5/2 & 0 \end{array}$$

$$\rightarrow \begin{array}{cccccc} Z & x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 0 & 0 & 0 & 3/2 & 1 \end{array}$$

- Coefficients of $x_3, x_4, x_5 = y_1^*, y_2^*, y_3^* = (0, 3/2, 1)$.
- For the managers, they may increase allocation of b_2 and b_3 by taking them from other productions if those $y_2^*(< 1.5)$ and $y_3^*(< 1)$.

Application of Duality Theory

- Primal Problem

$$
\begin{aligned}
Max \quad & Z = \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \\
& for \;\; i = 1, 2 \cdots m \\
and \quad & x_j \geq 0, j = 1, 2 \cdots n
\end{aligned}
$$

- Dual Problem

$$
\begin{aligned}
Min \quad & y_0 = \sum_{i=1}^{m} b_i y_i \\
s.t. \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_{j,} \\
& for \;\; j = 1, 2 \cdots n, \\
and \quad & y_i \geq 0, for \;\; i = 1, 2 \cdots m
\end{aligned}
$$

- In matrix form

$$\begin{aligned} &Max \quad Z = cx = \left[\cdots c \cdots\right] \begin{bmatrix} \vdots \\ x \\ \vdots \end{bmatrix} \\ &s.t. \quad Ax \leq b, \\ &and \quad x \geq 0 \end{aligned}$$

$$\begin{aligned} &Min \quad y_o = yb = \left[\cdots y \cdots\right] \begin{bmatrix} \vdots \\ b \\ \vdots \end{bmatrix} \\ &s.t. \quad yA \geq c \\ &and \quad y \geq 0 \end{aligned}$$

Weak Duality Property ($cx < yb$)

If x is a feasible solution for the primal problem and y is a feasible solution for the dual problem,then $cx \leq yb$.

Example

$$eg. \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad [y_1, y_2, y_3] = [1, 1, 2]$$

$$cx = [3, 5] \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 24 \qquad yb = [1, 1, 2] \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 52$$

$$cx < yb$$

Strong Duality Property ($cx^* = y^*b$)

- If x^* is the optimal solution for the primal problem, y^* is also an optimal solution for the dual problem, then $cx^* = y^*b$.
- Thus, the two properties imply that $cx < yb$ for feasible solutions if one or both of them are not optimal (i.e., $cx < yb$, $cx^* < yb$, or $cx < y^*b$), whereas equality holds when both are optimal.

Complementary-Solution Property

- At each iteration, the simplex method simultaneously produces a CPF solution x for the primal problem and a complementary solution y for the dual problem, where $cx = yb$.
- If x is not optimal for the primal, then y is not feasible for the dual problem.

Example

Example

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$cx = [3,5]\begin{bmatrix} 0 \\ 6 \end{bmatrix} = 30$$

$$cx = yb$$

$$[y_1, y_2, y_3] = [0, 5/2, 0]$$

$$yb = [0, 5/2, 0]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 30$$

- $cx = yb$
- x is not optimal and y is not feasible.

Complementary Optimal Solution Property

- At the final iteration, the simple method simultaneously finds an optimal solution x^* for the primal problem and a complementary optimal solution y^* for the dual problem, where $cx^* = y^*b$.
- y^* is feasible to the dual problem since x^* is optimal. In fact, y^* is also optimal to the dual.
- y_i^* are the shadow-prices for the primal problem.
- $y_i^* = \frac{dZ}{db_i}|_{x=x^*}$

Symmetry Property

- For any primal problem and its dual problem, all relationships between them must be symmetric, because the dual of the dual problem is the primal problem.
- Therefore, all the preceding properties hold regardless of which of the two problems is labeled as the primal problem.

Duality Theorem

The following are the only relationships between the primal and dual problem.

- If one problem has feasible solution and a bounded objective function (i.e it has an optimal solution), then so does the other problem. Both weak duality ($cx < yb$) and strong duality ($cx^* = y^* b$) properties are applicable here.

Duality Theorem

- If one problem has feasible solution and an unbounded objective function (i.e no optimal solution), then the other problem has no feasible solution.
- If one problem has no feasible solution, then the other problem has either no feasible solution or an unbounded objective function.

Application of Duality Theorem

- If $m > n$, then solving the dual may be easier (the constraints become less than the primal):
- If we find x and y (both are feasible) and if $cx = yb$, then we know we have found the optimal solution x^* and y^*. Even if they are not equal, $cx < yb$,if $yb - cx$ is small. We know x is close to x^*, then we may stop.
- Another important application is its use in the economic interpretation of the dual problem (shadow prices), and the resulting insight for analyzing the primal problem.
- y_i^*=shadow prices. $y_i^* = \dfrac{dZ}{db_i}\Big|_{x=x^*}$.

Economic Interpretation of Duality

- x_i =number of product j we want to produce.
- c_j =unit of profit from product j.
- Z = total profit.
- b_i =amount of resource of type i available.
- a_{ij} =amount of resource of type i needed by each unit of product j.

$$Max \quad Z = cx = \left[\cdots c \cdots\right] \begin{bmatrix} \vdots \\ x \\ \vdots \end{bmatrix} \qquad Min \quad y_o = yb = \left[\cdots y \cdots\right] \begin{bmatrix} \vdots \\ b \\ \vdots \end{bmatrix}$$

$$\begin{array}{llcll} s.t. & Ax \leq b, & \qquad & s.t. & yA \geq c \\ and & x \geq 0 & \qquad & and & y \geq 0 \end{array}$$

- $y_0 = b_1 y_1 + \cdots + b_m y_m$ =the current contribution to the profit by having b_i units of resource i available for the primal problem.
- $\sum_{i=1}^{m} a_{ij} y_i \geq c_j$ that the actual contribution to profit of the above mix of resources must be at least as much as they need by 1 unit of product j; otherwise the use of the resources mix is not good.

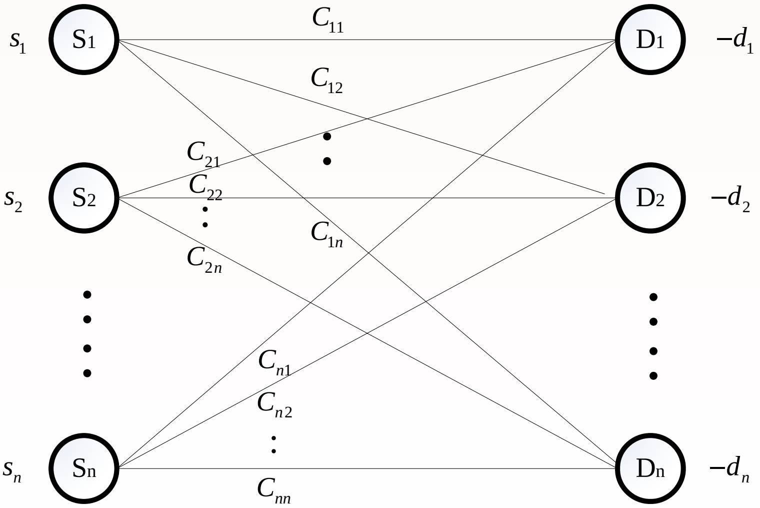
Complementary Slackness Property

- $y_i = 0$, if the corresponding slack variable is positive.
- $\frac{dZ}{db_i}|_x = 0$, if the slack variable for b_i is positive.
- $\sum_{i=1}^{m} a_{ij} y_i = c_j$ if $x_j > 0$. When the number of product j produced is positive, then the marginal profit $(\frac{dZ}{db_i}|_x = y_i)$ of the resource it consumes must equal to the unit profit of product j.

Example

- Optimal $x^* = (2, 6, 2, 0, 0), y_1^* = 0, y_2^* = 3/2, y_3^* = 1$.
- $[0, 3/2, 1] \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 5 \end{bmatrix}$
- Both constraints are equal since $x_1 > 0$, and $x_2 > 0$.

- Transportation Problem



- Transportation Problem

$$
\begin{aligned}
Min \quad & Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
s.t. \quad & \sum_{j=1}^{n} x_{ij} = s_i & for\ i = 1, 2 \cdots m \\
& \sum_{i=1}^{m} x_{ij} = d_j & for\ j = 1, 2 \cdots n \\
& x_{ij} \geq 0 & \forall i, j
\end{aligned}
$$

Remarks:

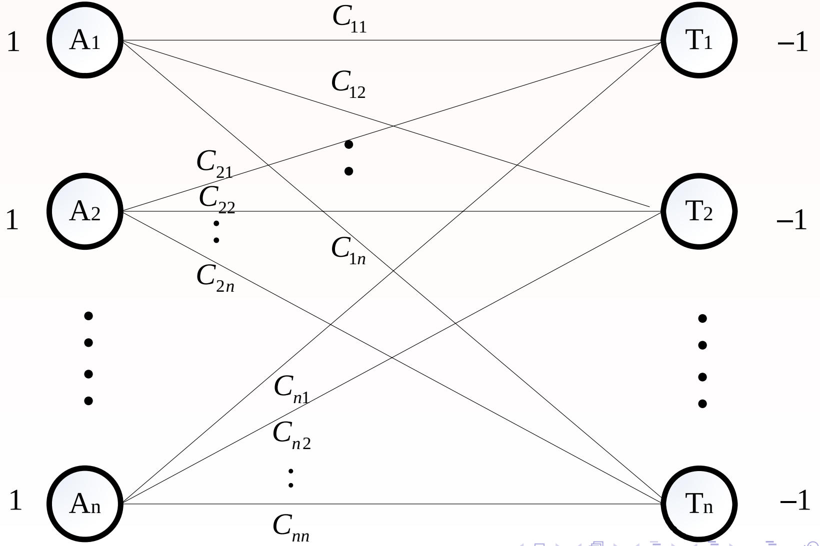
$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$ is assumed above.

If total supply $>$ total demand, then create dummy demand node D_0 with $c_{i0} = 0,\ i = 1, ..., m$.

If total supply $<$ total demand, then create dummy supply node S_0 with $c_{0j} = 0,\ j = 1, ..., n$.

Assignment Problem

- Assignment Problem



- Assignment Problem

$$
\begin{aligned}
Min \quad & Z = \textstyle\sum_i \sum_j c_{ij} x_{ij} \\
s.t. \quad & \sum_{j=1}^{n} x_{ij} = 1 && \forall i \\
& \sum_{i=1}^{m} x_{ij} = 1 && \forall j \\
& x_{ij} \geq 0 && x_{ij} = 1 \text{ or } 0
\end{aligned}
$$

Remarks:

$m = n$ is assumed above.

If $m > n$, then create dummy node T_0 with $c_{i0} = 0, \; i = 1, ..., m$.

If $m < n$, then create dummy node A_0 with $c_{0j} = 0, \; j = 1, ..., n$.

Network Analysis (special LP)

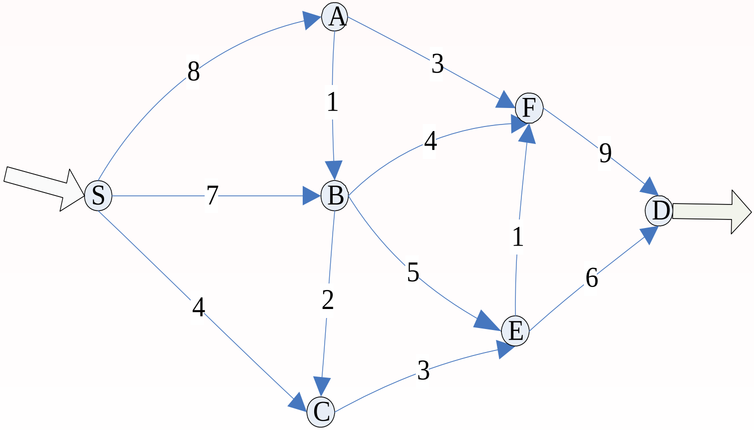
Topics

- Shortest-path problem.
- Minimum spanning tree.
- Maximum flow problem.
- Minimum cost flow problem.

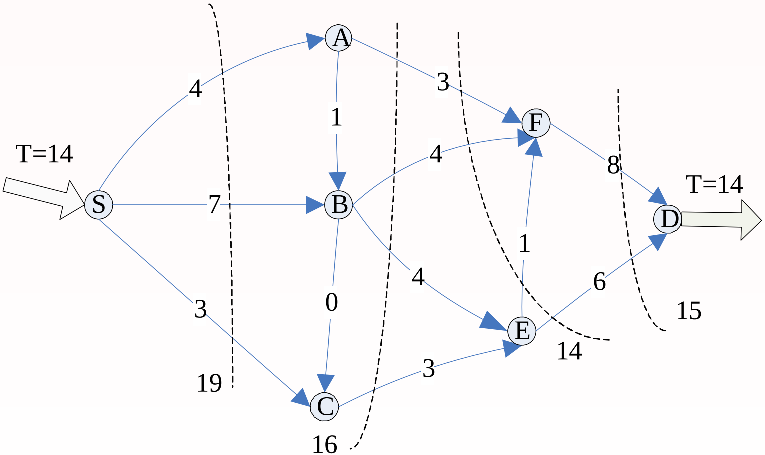
Max-Flow Min-Cut Theorem

- A cut may be defined as any set of directed arcs containing at least one arc from every directed path from source node to destination node.
- The cut value is the sum of the arc capacity of the cut.
- The theorem says: For any network with a single source node and destination node, the maximum feasible flow from S to D equals the minimum cut value for all cuts of the network.

Maximum Flow Problem (efficient solution method exists)



Optimal Solution



The Minimum Cost Flow Problem (very efficient solution method exists)

- It encompasses

$$\left.\begin{array}{l}\text{1. Transportation Problem.}\\ \text{2. Assignment Problem.}\\ \text{3. Shortest-path Problem.}\\ \text{4. Max Flow Problem.}\end{array}\right\}\begin{array}{l}\text{Special case of the}\\ \text{minimum cost flow problem}\end{array}$$

The Minimum Cost Flow Problem

- A network of n nodes, at least one source node (supply node) and one destination node (demand node).
- Find x_{ij} =amount of flow through link $i \to j$ to minimize the total cost.
- Given

$$\begin{aligned} c_{ij} &= \text{cost per unit flow for link } i \to j \\ u_{ij} &= \text{capacity of link } i \to j \\ b_i &= \text{net flow generated at node } i \end{aligned}$$

$$\begin{aligned} b_i &> 0 \text{ if } i \text{ is the source node} \\ b_i &< 0 \text{ if } i \text{ is the destination node} \\ b_i &= 0 \text{ if } i \text{ is the transit node} \end{aligned}$$

The Minimum Cost Flow Problem

$$
\begin{aligned}
\min \quad & Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\
s.t. \quad & \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = b_i \quad && \forall i \\
& 0 \le x_{ij} \le u_{ij} && \forall i, j
\end{aligned}
$$

The Minimum Cost Flow Problem (Solution Properties)

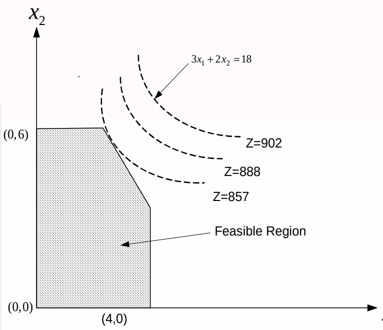
- Feasible solution property.
  - A necessary condition for a minimum cost flow problem to have any feasible solution is that $\sum_{i=1}^{n} b_i = 0$ ($i.e.$ total supply $=$ total demand). If not, create dummy source or dummy destination node with $c_{ij} = 0$.
- Integer solution property.
  - For minimum cost flow problem when all b_i and u_{ij} are integer, all the feasible solution from simplex method including the optimal x^* have the integer value.

Non-Linear Programming

$$
\begin{aligned}
\max \quad & f_0(x) \\
s.t. \quad & f_i(x) \le b_i \qquad\qquad i = 1, \cdots m \\
& x_1, x_2, \cdots x_n \ge 0
\end{aligned}
$$

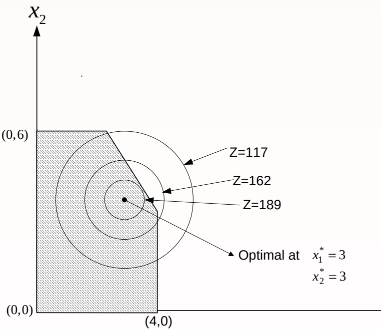
Example

$$
\begin{aligned}
\max \quad & Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2 \\
s.t. \quad & x_1 \le 4 \\
& 2x_2 \le 12 \\
& 3x_1 + 2x_2 \le 18 \\
& x_1 \ge 0 \\
& x_2 \ge 0
\end{aligned}
$$



Example

$$
\begin{aligned}
\max \quad & Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2 \\
s.t. \quad & x_1 \le 4 \\
& 2x_2 \le 12 \\
& 3x_1 + 2x_2 \le 18 \\
& x_1 \ge 0 \\
& x_2 \ge 0
\end{aligned}
$$



Necessary and Sufficient Conditions for Optimality

Problem	Necessary Condition for Optimality	Also sufficient if
one variable unconstrained	$\frac{df_0}{dx} = 0$	$f_0(x)$ is concave
Multi-variables unconstrained	$\frac{df_0}{dx_i} = 0 \quad (j = 1, \cdots, n)$	$f_0(x)$ is concave
General constrained problem	Karush-Kuhn-Tucker condition	$f_0(x)$ is concave and $f_i(x)$is concave $(i = 1, \cdots, m)$

Non-Linear Programming

$\max \; f_o(x)$ s.t. $f_i(x) \le b_i$ $i = 1, \cdots, m$ $x_j \ge 0$ $j = 1, \cdots, n$	$\max \; f_o(x)$ s.t. $f_i(x) - b_i \le 0$ $i = 1, \cdots, m$ $-x_j \le 0$ $j = 1, \cdots, n$	$\min \; f(x) = -f_o(x)$ s.t. $g_i(x) \le 0$ $i = 1, \cdots, m + n$ $x \in \mathbb{R}$

KKT Conditions

- Given a feasible $x^* \in \mathbb{R}^n$, if there exists $\lambda \in \mathbb{R}^m$,with $\lambda \geq 0$,such that

$$\begin{aligned} \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) &= 0 \qquad (1) \\ \text{and } \sum_{i=1}^{m} \lambda_i g_i(x^*) &= 0 \qquad (2) \end{aligned}$$

- Then x^* is a global optimal solutions for the primal problem.
- $\lambda' s$ are the Lagrange multipliers or dual variables.

KKT Conditions

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

$$\frac{\partial L}{\partial x_j} = 0 \Rightarrow \frac{\partial f(x)}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x)}{\partial x_j} = 0, j = 1, \cdots, n$$

In matrix form

$$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0 \ (1)$$

Duality

- Primal problem

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & g_i(x) \leq 0 \quad i = 1, \cdots, m \\ & x \geq 0 \end{aligned}$$

- Define $\theta(\lambda) = \inf\{x \geq 0 : f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}$.
- $\theta(\lambda)$ is called the Lagrangian dual function and is obtained by relaxing the constraints $g_i(x) \leq 0, i = 1, \cdots, m, \lambda \in \mathbb{R}^m, \lambda \geq 0$.

- $\theta(\lambda)$is a concave function on $\lambda \in \mathbb{R}^m, \lambda \geq 0$.
- If $\lambda \geq 0$,and if x satisfies the constraints of the primal problem, then $\theta(\lambda) \leq f(x)$(weak duality theorem).

Dual problem

- Dual problem $\max\limits_{\lambda \geq 0} \theta(\lambda)$.
- Strong duality theorem: $\theta(\lambda^*) = f(x^*)$.
- In addition, due to (2) of KKT, $\sum_{i=1}^{m} \lambda_i^* g_i(x^*) = 0$.

Relation to LP (Primal Problem)

- Primal problem

$$
\begin{aligned}
\min \quad & b^T x \\
s.t. \quad & Ax \geq c \\
& x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\theta(\lambda) &= \inf\{x \geq 0 : b^T x + \lambda^T (c - Ax)\} \\
&= \inf\{x \geq 0 : (b^T - \lambda^T A)x + \lambda^T c\} \\
&= \begin{cases} \lambda^T c & \text{for } b^T - \lambda^T A \geq 0 \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

Dual LP Problem

$$
\begin{aligned}
\max_{\lambda \geq 0} \quad & \theta(\lambda) = \max \lambda^T c \\
s.t. \quad & b^T - \lambda^T A \geq 0 \\
& \lambda \geq 0
\end{aligned}
$$

- Weak duality: $\lambda^T c < b^T x$.
- Strong duality: ${\lambda^*}^T c = b^T x^*$.
- Complementary slackness:

$$
\begin{aligned}
\lambda^T (c - Ax^*) &= 0 \\
(b^T - {\lambda^*}^T A) x^* &= 0
\end{aligned}
$$