Abstract — When the impulse response of a transfer system is long, it is difficult for the standard cross spectrum method to obtain an accurate estimate of the transfer function by applying the fast Fourier transform (FFT) with a finite length window. The bias error in the estimate is large especially around the resonant frequency of the transfer function. In this paper, therefore, we propose an alternative new method to obtain an accurate estimate of the transfer function. The delayed block transfer function is introduced to detect the components that are correlated to the signal in the input window but leak from the output window. Based on these transfer functions, the total characteristics of the transfer system are estimated accurately. In the latter half of this paper, we derive the theoretical expressions for the bias errors in the transfer functions estimated by the proposed and the standard methods. By thoroughly comparing the resultant expressions, the superiority and the usefulness of the proposed method are theoretically confirmed. Finally, the simulation experiments show the advantages of the proposed method.

I. INTRODUCTION

There has been a dramatic increase in spectrum estimation research activities, especially in the past two decades, since the digital fast Fourier transform (FFT) algorithm was introduced about 25 years ago [1]. The FFT has expanded the role of spectral estimation from research novelty to practical use. Although there are many disadvantages of such FFT-based results, the FFT is most commonly applied to unknown signals in many practical fields before using other parametric techniques because the expanded orthonormal functions employed in the FFT are predetermined and are independent of the signal.

Bingham et al. [4] discussed the computationally fastest way to estimate the power spectrum of a time series from the FFT performed directly on the weighted data set. Welch [5] proposed a method for the application of the FFT to the estimation of power spectra, which involves sectioning the recordings, taking modified periodograms of these sections, and averaging these modified periodograms. The weighted overlapped segment averaging is advocated by Nuttall and Carter [37], [41] to give stability and to minimize the impact of window sidelobes. Other references to the FFT and its application for power spectrum estimation may be found in Richards [2], Cochran et al. [3], Jenkins and Watt [6], Bertram [8], Glisson et al. [9], Cooley et al. [10], [11], Oppenheim and Shafer [19], Yuen [29], and Amin [56].

In the power spectrum estimation, leakage effects arising in the frequency domain due to the time domain windowing can be reduced by the selection of windows with nonuniform weighting. Bertram [12] provided a definitive description of the leakage problem. Harris [30] and Nuttall [43] have provided a good summary of the merits of various windows. Nuttall and Carter [46] presented a spectrum estimation method based on lag weighting. Other references include [16], [26], [39], [40], and [53].

As an application of these FFT-based spectrum estimation methods, the coherence function has been developed for a linear measure of causality in the transfer function between a pair of signals. Carter, Knapp, and Nuttall [17], [23] proposed the standard method to estimate the cross spectrum and the coherence function by partitioning the two signals into overlapping segments and computing the power spectrum of each segment via the FFT. The resultant power spectra are then averaged to reduce the bias and variance of the resulting estimates. The coherence function is effective in many fields as pointed out by Carter and Knapp [20], and it has been applied to system identification [17], measuring SNR and linear-to-nonlinear power ratio [17], and determining signal time delay [17], [18], [44]. Knapp and Carter [25] developed a maximum likelihood estimator (MLE) for determining time delay between signals based on the cross correlation, which is identical to one proposed by Hannan and Thomson [15]. A MLE for coherence is derived by Mohenkern [57]. Chan et al. [38] proposed a regression approach of the coherence function by solving the discrete Wiener-Hopf equation. Youn et al. [45], [47] introduced an adaptive approach based on Widrow’s LMS algorithm into the estimation of the coherence function for nonstationary signals. A tutorial review of work in coherence and time delay estimation was presented by Carter [50], [55]. Other references for the estimation of the coherence function or the cross spectrum using the FFT and their applications are found in Jenkins and Watt [6], Carter et al. [18], Talbot [22], Blake [28], Piersol [31], Seybert et al. [34], Barret [35], Chan [48], Cadzow [51], Cusani [58], and Mansour et al. [59].

The error analyses for their estimates are provided by Benignus [7], Bendat [33], Walker [42], Schmidt [49], Mathews et al. [52], and Gish et al. [54].

For the multiple input/output cases, Bendat [24], [27] provided methods to estimate the transfer function and the coherence function. Other references for the coherence function, partial coherence, and system identification are found in Jenk-
The cause of the bias error is that the signal in the output due to the insufficiency of the time window length is generated. Let us assume that the input signal and the measurement noise are stationary white noise and mutually uncorrelated. Let the average power of $x(n)$ and $n(n)$ be $\sigma_x^2$ and $\sigma_n^2$, respectively.

By applying an $MN$-point discrete Fourier transform to the input signal $x(n)$ and the output signal $y(n)$, both of which are windowed, where each window has $MN$ points in length, (1) is written as

$$Y(k) = H(k)X(k) + N(k)$$

where $k$ is an integer representing discrete frequencies, and the discrete spectrum $X(k)$ is given by

$$X(k) = \sum_{n=0}^{MN-1} x(n) \exp (-j2\pi kn/MN).$$

The least squares estimate, which we denote by $\hat{H}_{ls}(k)$, of the transfer function $H(k)$ in (2) is obtained by the cross spectrum method [13], [36] such as

$$\hat{H}_{ls}(k) = \frac{E[X^*(k)Y(k)]}{E[|X(k)|^2]}, \quad (k = 0, 1, \ldots, MN - 1)$$

where $E[.]$ and $*$ denote the ensemble average and a complex conjugate, respectively. For each frequency, the estimate $\hat{H}_{ls}(k)$ minimizes the average power $E[|N(k)|^2]$ of the noise component $n(n)$.

If the system response is long compared with the window length, however, the estimate $\hat{H}_{ls}(k)$ in (4) does not give accurate results. This is due to the following: The estimate $\hat{H}_{ls}(k)$ of (4) is obtained, assuming that the impulse response to every input pulse in a block is dropped at the end of the block as shown in Fig. 2(a). For example, the responses to the impulses $x(0)$, $x(MN/2)$, and $x(MN-1)$ are estimated so that they have, respectively $MN$-point, $MN/2$-point, and 1-point in length. These relations between $x(n)$ and $y(n)$ cannot be expressed by a linear system. In the least square estimate of $\hat{H}_{ls}(k)$ in (4), the nonlinear transfer system is approximated by a linear system. As a result, the bias error in the transfer function estimate $\hat{H}_{ls}(k)$ becomes larger around the resonant frequencies as the impulse response of the transfer system becomes longer. This bias error is caused even for the case of SNR = $\infty$ and for the case where the average number in (4) is infinite.

### III. A NEW METHOD OF ESTIMATING THE TRANSFER FUNCTION

From a different point of view, let us consider the reason the bias error happens in the transfer function, which is estimated by the standard method as follows: Let us divide the input signal $x(n)$ and the output signal $y(n)$ in a section with $MN$ points in length into two blocks $(x_1(n)$ and $x_2(n))$ and $(y_1(n)$ and $y_2(n))$, respectively, where each has $MN/2$ points in length as shown in Fig. 2(b). Using the spectra $X_1(k)$, $X_2(k)$,
Y_1(k), and Y_2(k) of these block signals, the averaged cross spectrum \( E[X^*(k)Y(k)] \) in the numerator of (4) is described by

\[
E[X^*(k)Y(k)] = E[X_1^*(k)Y_1(k)] + E[X_2^*(k)Y_2(k)] + E[X_1^*(k)Y_1(k)] + E[X_2^*(k)Y_2(k)].
\]

From the causality, the third term \( E[X_1^*(k)Y_1(k)] \) becomes zero. Let \( H_0(k) \) and \( H_1(k) \) be the transfer functions from one input block to the output block signal with the same timing and the one-block delayed signal, respectively. Thus, the cross spectrum \( E[X^*(k)Y(k)] \) is given by

\[
E[X^*(k)Y(k)] = E[|X(k)|^2](H_0(k) + H_1(k)) = E[|X(k)|^2](H_0(k) + H_1(k))
\]

that is, bias error is caused in the cross spectrum estimated in (4). From this simple example, it is found that it is significant to divide the input and output signals into short blocks and estimate the cross spectrum separately for each combination between the input blocks and the output blocks and then sum up the resultant transfer functions to obtain the total transfer function. In this section, based on the principle, a new method is proposed to decrease the bias error in \( H(n) \) of (4) and estimate an alternative transfer function \( H_{\text{alt}}(k) \) ranging from \( k = 0 \) to \( MN - 1 \) as described below.

Let us divide the \( MN \)-point length signal into \( M \) blocks, where each of consists of \( N \) consecutive samples, as shown in Fig. 3(a). Each block is identified by the block index. When the system response is long, the transmission system in Fig. 1 is represented by a multiple input/output system as shown in Fig. 3(b). Let an indexed \( H_i(k) \) be the transfer function (called the delayed block transfer function) from the input signal to the i-block-delayed-output signal, and let \( h_i(n) \) be the impulse response of \( H_i(k) \). The output signal \( y_m(n) \) in the mth block is the sum of the responses to each input signal in the preceding blocks and the measurement noise signal \( n_m(n) \) in mth block such as

\[
y_m(n) = \sum_{i=0}^{\infty} x_{m-i}(n) * h_i(n) + n_m(n) \tag{5}
\]

or by its expression in the frequency domain

\[
Y_m(k) = \sum_{i=0}^{\infty} H_i(k)X_{m-i}(k) + N_m(k)
\]

\[
= \sum_{i=0}^{\infty} Z_{m-i,m}(k) + N_m(k) \tag{6}
\]

where \( X_m(k), Y_m(k), \) and \( N_m(k) \) are, respectively, the spectra of the zero-padded \( MN \)-length signals \( x_m(n), y_m(n), \) and \( n_m(n) \) of the original \( N \)-length signals \( x_m(n), y_m(n), \) and \( n_m(n) \) in the mth block, where each is obtained as

\[
x_m(n) = \begin{cases} x(n), & \text{if } n = 0, 1, \ldots, N-1; \\ 0, & \text{if } n = N, N+1, \ldots, MN-1. \end{cases} \tag{7}
\]

From Fig. 3(b), the spectrum \( Z_{m-i,m}(k) \) of the response of \( H_i(k) \) to \( x_{m-i}(n) \) is defined as follows:

\[
Z_{m-i,m}(k) = H_i(k)X_{m-i}(k). \tag{8}
\]

The noise term \( N_m(k) \) in (6) can be rewritten as

\[
N_m(k) = Y_m(k) - \sum_{i=0}^{\infty} H_i(k)X_{m-i}(k).
\]

The expectations of the product of \( N_m(k) \) and its complex conjugate \( N_m^*(k) \) give the average noise power \( P_N(k) \) at the
Using the relation (11), \( V(k) \) is

\[
V(k) = \sum_{i=0}^{M-1} \left( \sum_{n=0}^{N-1} v_i(n) \exp\left(-j2\pi \frac{kn}{MN}\right) \right) \exp\left(-j2\pi \frac{ki}{M}\right)
\]

The term in the \{ \} brackets of this equation is equal to the spectrum \( V_i(k) \), \( k = 0, 1, \ldots, MN - 1 \), which is obtained by applying the \( MN \)-point FFT to the \( MN \)-point zero-padded signal \( v_i(n) \) defined by

\[
v_i(n) = \begin{cases} 
  v_i(n), & \text{if } n = 0, 1, \ldots, N - 1; \\
  0, & \text{if } n = N, N + 1, \ldots, MN - 1.
\end{cases}
\]  

Using the spectrum \( V_i(k) \), the total spectrum \( V(k) \) for the \( MN \)-point signal \( v(n) \) is described by

\[
V(k) = \sum_{i=0}^{M-1} V_i(k) \exp\left(-j2\pi \frac{ki}{M}\right).
\]  

In the relation of (13), by replacing \( V_i(k) \) and \( V(k) \), respectively, by the delayed block transfer function \( \tilde{H}_i(k) \) and the total transfer function \( \tilde{H}_{bk}(k) \), the estimate is given by

\[
\tilde{H}_{bk}(k) = \sum_{i=0}^{M-1} \tilde{H}_i(k) \exp\left(-j2\pi \frac{ki}{M}\right).
\]

Thus, the estimate \( \tilde{H}_{bk}(k) \) of the total transfer function is obtained from the \( M \) spectrum estimates \( \{ \tilde{H}_i(k) \} \) of the delayed block transfer function in (10).

IV. THEORETICAL DERIVATIONS FOR THE TRANSFER FUNCTION ESTIMATES

To confirm the accuracy improvement in the estimate of \( \tilde{H}_{bk}(k) \) in (14), the estimates of \( \tilde{H}_{iij}(k) \) in (4) and \( \tilde{H}_{bk}(k) \) in (14) and their bias errors are theoretically derived in this section. To begin with, the characteristics of the true transfer function are defined in Section IV-A. Then, after deriving the theoretical expression for the expectations of the cross spectrum and power spectrum in Section IV-B, the estimates of \( \tilde{H}_{iij}(k) \) and \( \tilde{H}_{bk}(k) \) are theoretically derived in Sections IV-C and D, respectively.

A. Definition of the Characteristics of a Transfer Function

Let us assume that the transfer system is a rational transfer function of order \( p \) and the impulse response \( h(n) \) is described by

\[
h(n) = \sum_{i=1}^{p} C_i p_i^i
\]

where \( \{ p_i \} \) are the poles of the transfer system (\( |p_i| < 1 \)), and \( \{ C_i \} \) are the complex coefficients. By defining

\[
z_0 = \exp\left(-j2\pi \frac{1}{MN}\right)
\]  

...
the $MN$-point Fourier transform of the impulse response $h(n)$
is obtained by

$$H(k) = \sum_{n=0}^{MN-1} h(n) \exp\left(-j2\pi \frac{kn}{MN}\right)$$

$$= \sum_{i=1}^{P} C_i \sum_{n=0}^{MN-1} \left(p_i z_0^k\right)^n$$

$$= \sum_{i=1}^{P} C_i \frac{1 - \left(p_i z_0^k\right)^{MN}}{1 - p_i z_0^k},$$

(17)

If the length of the FFT increases to the infinite, that is,
$M \to \infty$, the true transfer function $H_\infty(k)$ of $H(k)$ in (17)
is given by

$$H_\infty(k) = \sum_{i=1}^{P} C_i \frac{1}{1 - p_i z_0^k}. \quad \text{(18)}$$

B. Theoretical Derivations for the Cross Spectrum

Let $F_{om}(k; q)$ denote the $k$th spectrum of the signal $h(n + mN - q)$
in the $m$th block of the response of the transfer system to the impulse
$h(n - q)$ at a time $q$ in the 0th block as shown in Fig. 4(a). For the case of $m \geq 1$, by applying
the $MN$-point FFT to the zero-padded impulse response $h'(n + mN - q)$

$$h'(n + mN - q) = \begin{cases} 
  h(n + mN - q), & \text{if } n = 0, 1, \ldots, N - 1; \\
  0, & \text{if } n = N, N + 1, \ldots, MN - 1 
\end{cases}$$

and substituting (15) into the resultant spectrum, the spectrum
$F_{om}(k; q)$ is given by

$$F_{om}(k; q) = \sum_{i=1}^{P} C_i p_i^{mN-q} \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k}. \quad \text{(19)}$$

For the case of $m = 0$, by applying the $MN$-point FFT to the
zero-padded $(N - q)$-length impulse response $h'(n - q)$

$$h'(n - q) = \begin{cases} 
  h(n - q), & \text{if } n = q, q + 1, \ldots, N - 1; \\
  0, & \text{otherwise} 
\end{cases}$$

the spectrum $F_{00}(k; q)$ is obtained by

$$F_{00}(k; q) = \sum_{i=1}^{P} C_i p_i^{N-q} \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k}, \quad \text{(for } m = 0) \quad \text{(20)}$$

where the derivations of these spectrum $F_{om}(k; q)$ and
$F_{00}(k; q)$ in (19) and (20) are described in Appendix A.

Let $z_{om}(n)$ ($m = 0, 1, \ldots$) denote the $N$-point signal in the
$m$th block of the response of the transfer system to the input
signal $x_0(n)$ in the 0th block as shown in Fig. 4(b). Using the
expression for the $F_{00}(k; q)$ of (19) and (20), the $MN$-point
spectrum $Z_{om}(k)$ of $z_{om}(n)$ is obtained by

$$Z_{om}(k) = \sum_{q=0}^{N-1} x_0(q) F_{00}(k; q) \quad \text{(21)}$$

where $m = 0, 1, 2, \ldots$. By substituting (19) and (20) into (21)

$$Z_{om}(k) = \sum_{q=0}^{N-1} x_0(q) \frac{p_i p_i^{N-q} \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k}}{1 - p_i z_0^k} \sum_{q=0}^{N-1} x_0(q) p_i^{q}$$

$$= \sum_{i=1}^{P} C_i p_i^{N-q} \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k} \sum_{q=0}^{N-1} x_0(q) p_i^{q}$$

$$= \sum_{i=1}^{P} C_i \sum_{q=0}^{N-1} x_0(q) \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k}$$

$$= \sum_{i=1}^{P} \sum_{q=0}^{N-1} x_0(q) \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k} \quad \text{(m \geq 1); (22)}$$

$$Z_{00}(k) = \sum_{q=0}^{N-1} x_0(q) \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k} \quad \text{(23)}$$

Letting $G_{om}(k)$ be the expectations of the cross spectrum
between $X_0(k)$ and $Z_{om}(k)$ in (22) and (23), $G_{om}(k)$ is given by
\[ G_{0m}(k) = E[X_0^*(k)Z_{0m}(k)] = N\sigma_x^2(k) \sum_{i=1}^{p} C_i \frac{1 - (p\bar{z}_0)^k}{1 - p\bar{z}_0} \left[ 1 + \frac{1 - (p\bar{z}_0)^N}{N\{1 - (p\bar{z}_0)^{-1}\}} \right], \quad (m \geq 1) \]

(24)

and for the case of \( m = 0 \)
\[ G_{00}(k) = E[X_0^*(k)Z_{00}(k)] = N^2\sigma_x^2(k) \sum_{i=1}^{p} C_i \left[ 1 + \frac{1 - (p\bar{z}_0)^N}{N\{1 - (p\bar{z}_0)^{-1}\}} \right]. \]

(25)

where these derivations of the cross spectrum \( G_{0m}(k) \) and \( G_{00}(k) \) in (24) and (25) are described in Appendix B.

On the other hand, the expectations of the power spectrum obtained by the \( MN \)-point FFT of the zero-padded input signal \( x(n) \), which consists originally of \( N \) points, is calculated by
\[ E[|X(k)|^2] = E\left[ \sum_{n=0}^{N-1} x_0^*(n)x_0(n) \right] \sum_{q=0}^{N-1} x_0(q) \exp(-j2\pi \frac{kq}{MN}) \]
\[ = \sum_{n=0}^{N-1} \sum_{q=0}^{N-1} E[x_0^*(n)x_0(q)] \exp(-j2\pi \frac{k(q-n)}{MN}). \]

Since \( E[x_0^*(n)x_0(q)] = \sigma_x^2 \delta(n-q) \), the expectations of the power spectrum is
\[ E[|X(k)|^2] = N\sigma_x^2(k) \sum_{n=0}^{N-1} 1 = N^2\sigma_x^2(k). \]

(26)

From the ratio of the average cross spectrum \( G_{0m}(k) \) in (24) or \( G_{00}(k) \) in (25) to the average power spectrum of \( x(n) \) in (26), the estimates \( \tilde{H}_i(k) \) in (10) for the delayed transfer function are obtained by
\[ \tilde{H}_i(k) = \frac{G_{0i}(k)}{N^2\sigma_x^2(k)}. \quad \text{for } i = 0, 1, 2, \ldots, 27 \]

C. Transfer Function Estimates \( \tilde{H}_{all}(k) \) for the Standard Cross Spectrum Method

As shown previously in (4), \( \tilde{H}_{all}(k) \) denotes the estimate of the transfer function obtained by applying the \( MN \)-point FFT to the \( M \)-block signals \( x(n) \) and \( y(n) \), where each block consists of \( N \) points. The estimate \( \tilde{H}_{all}(k) \) is rewritten by
\[ \tilde{H}_{all}(k) = E[X_*(k)Y(k)] / E[|X(k)|^2]. \]

(4)

As described in Appendix C, the signals \( x(n) \) and \( y(n) \) are divided into \( M \)-block signals \( x_i(n) \) and \( y_i(n) \) (i = 0, 1, 2, \ldots, \( M-1 \)), respectively. Using the power spectrum \( |X_i(k)|^2 \) and the cross spectrum \( G_{1-i}(k) = E[X_i^*(k)Y_i(k)] \) of the spectrum \( X_i(k) \) and \( Y_i(k) \) of the resultant signals \( x_i(n) \) and \( y_i(n) \), the estimate \( \tilde{H}_{all}(k) \) in (4) is described as follows:
\[ \tilde{H}_{all}(k) = \frac{\sum_{m=0}^{M-1} M - m G_{m}(k)}{MN^2\sigma_x^2(k)} \exp(-j2\pi \frac{km}{M}). \]

(28)

The term \( G_{m}(k)/N^2\sigma_x^2(k) \) is equal to the \( \tilde{H}_m(k) \) in (10). Thus, using the estimate \( \tilde{H}_m(k) \) of the delayed block transfer function, the transfer function estimate \( \tilde{H}_{all}(k) \) is obtained by
\[ \tilde{H}_{all}(k) = \sum_{m=0}^{M-1} M - m \tilde{H}_m(k) \exp(-j2\pi \frac{km}{M}). \]

(29)

As described in Appendix C, by rearranging after substituting \( \tilde{H}_m(k) \) in (27), the cross spectra \( G_{0m}(k) \) in (24) and (25), and \( z_0 \) in (16) into (29)
\[ \tilde{H}_{all}(k) = \sum_{i=1}^{p} C_i \left[ 1 + \frac{1 - (p\bar{z}_0)^N}{N\{1 - (p\bar{z}_0)^{-1}\}} \right] \frac{1 - (p\bar{z}_0)^MN}{M}. \]

(30)

Since the first term \( \sum_{i=1}^{p} C_i \frac{1}{1 - p\bar{z}_0} \) of \( \tilde{H}_{all}(k) \) is equal to the true transfer function \( H_{\infty}(k) \) in (18), the remaining term shows the bias error \( \tilde{H}_{all}(k) \). Thus, the bias error \( \Delta \tilde{H}_{all}(k) \) is given by
\[ \Delta \tilde{H}_{all}(k) = \tilde{H}_{all}(k) - H_{\infty}(k) \]
\[ = \sum_{i=1}^{p} C_i \left[ 1 + \frac{1 - (p\bar{z}_0)^N}{N\{1 - (p\bar{z}_0)^{-1}\}} \right] \frac{1 - (p\bar{z}_0)^MN}{M}. \]

(31)

D. Transfer Function Estimate \( \tilde{H}_{bk}(k) \) for the Proposed Method

Let us consider the theoretical derivation for the estimate \( \tilde{H}_{bk}(k) \) of the total transfer system in (14) using the definition of the impulse response \( h(n) \) in (15) as follows:

By substituting \( H_i(k) \) of (27) into (14) and using the definition of \( z_0 \) in (16), the estimate \( \tilde{H}_{bk}(k) \) is obtained by
\[ \tilde{H}_{bk}(k) = \frac{1}{N^2\sigma_x^2(k)} \left[ G_{00}(k) + \sum_{m=1}^{M-1} G_{0m}(k)z_0^{kmN} \right]. \]

By arranging after substituting the cross spectra \( G_{00}(k) \) of (24) and \( G_{0m}(k) \) of (25) into this equation as described in Appendix D, the estimate \( \tilde{H}_{bk}(k) \) of the total transfer system obtained by the proposed method is given by
\[ \tilde{H}_{bk}(k) = \sum_{i=1}^{p} C_i \left[ 1 + \frac{(p\bar{z}_0)^k}{N\{1 - (p\bar{z}_0)^{-1}\}} \right] \frac{1 - (p\bar{z}_0)^MN}{M}. \]

(32)

Let us consider the physical meaning of \( \tilde{H}_{bk}(k) \). Let \( \tilde{H}_0(k) \) be the transfer function estimate obtained by applying the standard method to the \( N \)-point input and output signals. By
substituting $N$ into $MN$ of the segment length in $\hat{H}_{all}(k)$ of (30), the transfer function estimate $\tilde{H}_{0}(k)$ is given by

$$\tilde{H}_{0}(k) = \sum_{i=1}^{P} C_i \frac{1}{1 - p_i z^{-1}} \left[ 1 + \frac{1}{N} \left( 1 - (p_i z^{-1})^{MN} \right)^{-1} \right].$$

The second term in the right-hand side shows the bias error. In the estimate $\hat{H}_{all}(k)$ of (30), which is given when the window length $N$ of $\hat{H}_{0}(k)$ is increased to $MN$, the bias error of $\hat{H}_{0}(k)$ is multiplied by $1 - (p_i z^{-1})^{MN} < 1$, that is, the bias error is decreased in inverse proportion to the window length. In the estimate $\hat{H}_{bk}(k)$ of (32), however, the bias error of $\hat{H}_{0}(k)$ is multiplied by exponentially decaying term $(p_i z^{-1})^{(M-1)N} < 1$. Thus, a more accurate estimate of the transfer function is obtained by the proposed method.

As the same manner in (31), the first term $\sum_{i=1}^{P} C_i \frac{1}{1 - p_i z^{-1}}$ of $\Delta \hat{H}_{bk}(k)$ in (32) is equal to the true transfer function $H_{bk}(k)$ in (18), and the remaining term expresses the bias error $\Delta \tilde{H}_{bk}(k)$ in the estimate $\tilde{H}_{bk}(k)$. Therefore

$$\Delta \tilde{H}_{bk}(k) = \tilde{H}_{bk}(k) - H_{bk}(k)$$

$$= \sum_{i=1}^{P} C_i \frac{1}{1 - p_i z^{-1}} \left[ 1 + \frac{1}{N} \left( 1 - (p_i z^{-1})^{MN} \right)^{-1} \right]$$

$$\times (p_i z^{-1})^{(M-1)N} \left( 1 - (p_i z^{-1})^{MN} \right).$$

(33)

If the number $M$ of the blocks and the length $N$ of each block in (33) are substituted by 1 and $MN$, respectively, as shown in the extreme left of Fig. 5(a), the proposed method coincides with the standard cross spectrum method, and in this case, $\Delta \tilde{H}_{bk}(k)$ in (33) coincides exactly with $\Delta \hat{H}_{all}(k)$ in (31).

Alternatively, when the number $M$ of the blocks and the length $N$ of each block in (33) are substituted by $MN$ and 1, respectively, as shown in the extreme right of Fig. 5(a), each block consists of only one point, and the number of the blocks becomes equal to the number of total points of the signals. In this case, the spectrum $Y_m(k)$ of $m$th block, which consists of one-point signal $y(m)$, is given by

$$\tilde{Y}_m(k) = \sum_{n=0}^{N-1} y(n+m) \exp(-j2\pi \frac{kn}{MN})$$

$$= y(m).$$

Thus, the delayed block transfer function $H_i(k)$ in (10) is obtained as

$$\hat{H}_i(k) = \frac{E[X_m(*)(k)y_m(k)]}{E[X_m(*)(k)]^2} = \frac{E[x(m-i)^*y(m)]}{E[x(m-i)]^2} = \widetilde{R}_{xy}(i)$$

where $\widetilde{R}_{xy}(i)$ denotes the estimates of the normalized correlation function between $x(n)$ and $y(n+i)$. From (14), the total transfer function, which is denoted by $\hat{H}_{BT}(k)$, is equal to the $MN$-point Fourier transform of the correlation function such as

$$\hat{H}_{BT}(k) = \hat{H}_{bk}(k)_{|_{M=M,N,N-1}}$$

$$= \sum_{i=0}^{M-1} \widetilde{R}_{xy}(i) \exp(-j2\pi \frac{ki}{MN})$$

(34)

which corresponds to the spectrum estimates obtained by the Blackman–Tukey(B-T) method [19]. By substituting $MN$ and 1 into $M$ and $N$ of (33), respectively, the bias error $\Delta \hat{H}_{BT}(k)$ in the estimate $\hat{H}_{BT}(k)$ is given by

$$\Delta \hat{H}_{BT}(k) = \Delta \hat{H}_{bk}(k)_{|_{M=M,N,N-1}}$$

$$= \sum_{i=1}^{P} C_i \frac{1}{1 - (p_i z^{-1})^{MN} \left( 1 - (p_i z^{-1})^{(M-1)N} \right)^{-1}} \times (p_i z^{-1})^{MN-1}. \quad \text{(35)}$$

V. ACCURACY COMPARISON BASED ON THE THEORETICALLY DERIVED EQUATIONS

By comparing the difference of the bias error between $\Delta \hat{H}_{all}(k)$ in (31) and $\Delta \hat{H}_{bk}(k)$ in (33), it is clear that there is difference between them only in the third terms $(p_i z^{-1})^{(M-1)N} \left( 1 - (p_i z^{-1})^{MN} \right)$ in $\Delta \hat{H}_{bk}(k)$ and $\frac{1}{N} \left( 1 - (p_i z^{-1})^{MN} \right) \Delta \hat{H}_{all}(k)$. When the actual length of the signal involved in the zero-padded signal is equal to $N$, let us define the ratio $\eta(k; i, N)$ of $\Delta \hat{H}_{bk}(k)$ to $\Delta \hat{H}_{all}(k)$ for each order $i$ of the impulse response $C_i p_i^k$ of the poles in the transfer system as follows:

$$\eta(k; i, N) \equiv \frac{\Delta \hat{H}_{bk}(k)}{\Delta \hat{H}_{all}(k)}$$

$$= M (p_i z^{-1})^{(M-1)N} \left( 1 - (p_i z^{-1})^{MN} \right)$$

$$= MN (p_i z^{-1})^{MN} \left( 1 - (p_i z^{-1})^{MN} \right)^{-N-1}. \quad \text{(36)}$$
The first term $\frac{MN \sum_{k=1}^{M} \alpha(k)}{1-(\mu x)^{2} MN}$, which we denote by $D(k;i)$ hereafter, depends not on $N$ but on the total length $MN$ of the impulse response, which is assumed to be a constant value.

Since the bias error is large especially around the resonant frequency, let us evaluate $\eta(k;i,N)$ at and near the resonant frequency as follows: At the resonant frequency $k_{i} = MN \mu 2\pi / $ of the $i$th pole, the complex term $\alpha(k;i)$ becomes a real value, which we denote by $a$ ($0 < a < 1$).

Thus, the ratio $\eta(k;i,N)$ and $D(k;i)$ become real at $k = k_{i}$. In this case, the ratio $\eta(k;i,N)$ at $k = k_{i}$ is given by

$$\eta(k;i,N) = D(k;i) a^{-N} - 1 \quad (37)$$

where $D(k;i) > 0$. Since the partial derivative of $a^{-N} = \exp(-N \ln a)$ with respect to $N$ is equal to $-a^{-N} \ln a$, the partial derivative of $\eta(k;i,N)$ with respect to $N$ is given by

$$\frac{\partial \eta(k;i,N)}{\partial N} = D(k;i) \left\{ - \frac{a^{-N} - a^{-N} \ln a}{N} \right\}$$

$$= D(k;i) a^{-N} \left( a^{-N} - 1 \right)$$

Since the second term $(a^{-N} - 1)$ is positive for $0 < a^{-N} < 1$, the gradient $\frac{\partial \eta(k;i,N)}{\partial N}$ of $\eta(k;i,N)$ is positive for $0 < a^{-N} < 1$. Thus, $\eta(k;i,N)$ increases monotonically as $N$ becomes larger in the range of $1 \leq N \leq MN$ as illustrated in Fig. 5(b). When $N$ is equal to $MN$, that is, $M = 1$, $\Delta \bar{H}_{b}(k)$ becomes equal to $\Delta \bar{H}_{s}(k)$ as described previously. In this case, $\eta(k;i,N = MN) = 1$. Therefore, the bias error $\Delta \bar{H}_{b}(k)$ in the proposed method is always less than or equal to $\Delta \bar{H}_{s}(k)$ in the standard cross spectrum method at the resonant frequency, that is

$$\Delta \bar{H}_{b}(k) \geq \Delta \bar{H}_{s}(k) \geq \Delta \bar{H}_{BT}(k). \quad (38)$$
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Fig. 8. Phase characteristics of the bias error in the estimate $\tilde{H}(k)$ by the proposed method in (32) for different five combinations of the number $M$ of the blocks and the length $N$ of a block. The dotted line shows the bias error $\Delta \tilde{H}(k)$ in the estimate $\tilde{H}(k)$ of the standard cross spectrum method in (30): (a) $N = 64$, $M = 1$; (b) $N = 32$, $M = 2$; (c) $N = 16$, $M = 4$; (d) $N = 8$, $M = 8$; (e) $N = 1$, $M = 64$.

VI. COMPUTER SIMULATION EXPERIMENTS

In order to illustrate the advantage of the proposed method to estimate the transfer function $H(k)$ or its impulse response $h(n)$, we choose the example of the fourth order all-pole transfer model $H_{\infty}(k)$, where poles are $0.996 \exp(\pm 2\pi \cdot 65/360)$ and $0.997 \exp(\pm 2\pi \cdot 80/360)$ as shown in Fig. 11(1-b). The input signal $x(n)$ in (1) is assumed to be white noise, and the output signal $y(n)$ is contaminated by measured white noise $n(n)$, which is uncorrelated with the driving series $x(n)$. The SNR equals 15 dB, and 32,768 points are generated for each of the signals $x(n)$ and $y(n)$. The generated two signals $x(n)$ and $y(n)$ are then divided into 128 sections, each of which has length $(MN)$ equal to 256 points.

Fig. 9. Relation between the number $M$ of blocks used in the proposed method and the magnitude of the bias error $\Delta \tilde{H}(k)$ in (33) at the resonant frequency $k_1 = \frac{65}{360} \times MN \approx 9$ under the condition that the total length $MN$ is always equal to 64. $|p_1| = 0.95$. The dotted line shows the bias error in the estimate $\tilde{H}(k)$ of the standard cross spectrum method: (a) $|p_1| = 0.95$; (b) $|p_1| = 0.9$; (c) $|p_1| = 0.8$.

Fig. 10. Relation between the number $M$ of blocks used in the proposed method and the phase of the bias error $\Delta \tilde{H}(k)$ in (33) near the resonant frequency $k_1 = 10$ under the condition that the total signal length $MN$ is always equal to 64. $|p_1| = 0.95$. The dotted line shows the bias error in the estimate $\tilde{H}(k)$ of the standard cross spectrum method: (a) $|p_1| = 0.95$; (b) $|p_1| = 0.9$; (c) $|p_1| = 0.8$.

response $h_{\infty}(n)$ of $H_{\infty}(k)$ is shown for the same length in Fig. 11(1-a). Assuming that the true length of the impulse response is not known $a\ a\ priori$, the impulse response is estimated for the length $MN = 256$ points in the following experiments for both the standard method and for the proposed method.
Fig. 11. (1) Magnitude characteristics of the fourth-order all-pole transfer model \( H_\infty(k) \) employed in the computer simulation in Section VI and its impulse response \( h_\infty(n) \); (2) transfer function estimate \( \hat{H}_{\text{all}}(k) \) and its impulse response \( \hat{h}_{\text{all}}(n) \) obtained by the standard method described in Section II.

Fig. 11(2) shows the transfer function estimates \( \hat{H}_{\text{all}}(k) \) and the impulse response estimate \( \hat{h}_{\text{all}}(n) \), which are obtained by applying the standard method in Section II to the signals \( x(n) \) and \( y(n) \) in each segment with length \( MN = 256 \), which is divided above. The number of the nonoverlapping average operation \( E[.] \) in (4) is equal to 128 times. As described previously in Section II, the estimate \( \hat{H}_{\text{all}}(k) \) is obtained so that the impulse response to every input impulse in a segment is truncated at the end of the segment. In the magnitude of the transfer function estimate \( \hat{H}_{\text{all}}(k) \) in Fig. 11(2-b), a spectrum zero appears due to the truncation. The resultant impulse response estimate \( \hat{h}_{\text{all}}(n) \) in Fig. 11(2-a) converges more rapidly than the true characteristic \( h_\infty(n) \) in Fig. 11(1-a). From these results, the true length of the impulse response \( h_\infty(n) \) cannot be estimated by the standard method.

Figs. 12 and 14 show the results obtained by the method proposed in Section III. For the results in Figs. 12 and 14(1), each segment with length \( MN = 128 \) is divided into nonoverlapping 8 blocks, each of which has 32 points in length; that is, in (7), (10), and (14), the number \( M \) of the blocks and the length \( N \) of each block are 8 and 32, respectively. Fig 12 shows the impulse response estimates \( \hat{h}_i(n) \), \( i = 0, 1, 2, \ldots, 8 \), which are obtained from the discrete Fourier transform (DFT) of \( \hat{H}_i(k) \cdot \exp(-j2\pi \cdot ki/M) \) in the right-hand side of (14). The impulse response \( \hat{h}_i(n) \) shows the transfer characteristics from the zero-padded input signal \( x_m^{(\infty)}(n) \) in the \( m \)-th block with length \( N \) to the zero-padded \( i \)-block delayed output signal \( y_{m+i}(n) \). For the estimate \( \hat{h}_0(n) \) in Fig. 12(a), the lengths \( \ell \) of the shortest and longest paths from the input block to the output block are 0 and \( N - 1 \), respectively, as shown in Fig. 13(a). However, for \( \hat{h}_1(n) \) in Fig. 12(b), the lengths \( \ell \) of the shortest and longest paths are 1 and \( 2N - 1 \), respectively, as shown in Fig. 13(b). Thus, the duration time of the resultant impulse response estimate is \( 2N - 1 \) for \( \hat{h}_1(n) \), \( \hat{h}_2(n), \ldots, \hat{h}_8(n) \) as shown in Figs. 12(b)-(i). For the estimates \( \hat{h}_8(n) \) in Fig. 12(i), the later half of the estimates is shifted to the beginning of the estimates due to the aliasing in the above DFT operation of \( \hat{H}_i(k) \cdot \exp(-j2\pi \cdot ki/M) \).

Thus, by eliminating the beginning part of the estimate \( \hat{h}_i(n) \) and summing up the resultant estimates \( \hat{H}_i(k) \cdot \exp(-j2\pi \cdot ki/M) \) for \( i = 0, 1, \ldots, 8 \) in (14), the transfer function estimates \( \hat{H}_{\text{all}}(k) \), and its impulse response \( \hat{h}_{\text{all}}(n) \) is obtained as shown in Figs. 14(1-b) and (1-c), respectively. The impulse response estimates \( \hat{h}_{\text{all}}(n) \) in Fig. 14(1-a) almost...
The kth spectrum $F_{om}(k; q)$ in Section IV-B

The kth spectrum $F_{om}(k; q)$ of the signal $h(n + mN - q)$ in the $m$th block of the response of the transfer system to the impulse $b(n - q)$ at a time $q$ in the 0th block as shown in Fig. 4(a) is obtained as follows: For the case of $m \geq 1$, by applying the $MN$-point FFT to the zero-padded impulse response $h'(n + mN - q)$ defined by

$$h'(n + mN - q) = \begin{cases} h(n + mN - q), & \text{if } n = 0, 1, \ldots, N - 1; \\ 0, & \text{if } n = N, N + 1, \ldots, MN - 1 \end{cases}$$

the spectrum $F_{om}(k; q)$ is given by

$$F_{om}(k; q) = \sum_{n=0}^{N-1} h(n + mN - q)z_0^n, \quad (m \geq 1) \quad (A.1)$$

where $z_0 = \exp(-j2\pi/MN)$ as defined in (16). For the case of $m = 0$, by applying the $MN$-point FFT to the zero-padded
(N - q)-length impulse response \( h'(n - q) \)

\[
h'(n - q) = \begin{cases} 
  h(n - q), & \text{if } n = q, q + 1, \ldots, N - 1; \\
  0, & \text{otherwise}
\end{cases}
\]

the spectrum \( F_{00}(k; q) \) is obtained by

\[
F_{00}(k; q) = \sum_{n=q}^{N-1} h(n - q)z_0^{kn}.
\] (A.2)

By substituting \( h(n) \) defined in (15) into (A.1) and (A.2), the spectra \( F_{0m}(k; q) \) and \( F_{00}(k; q) \) are, respectively, described by

\[
F_{0m}(k; q) = \sum_{n=m}^{N-1} \sum_{q=1}^{P} C_{pi} x_0^{n+3mN+q-k} z_0^{kn}.
\]

\[
= \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} (p_i z_0^k)^q \sum_{n=m}^{N-1} z_0^{-kn} p_i^n
\]

\[
= \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k} (n \geq 1);
\] (A.3)

\[
F_{00}(k; q) = \sum_{n=q}^{N-1} \sum_{q=1}^{P} C_{pi} x_0^{n+3mN+q-k} z_0^{kn}
\]

\[
= \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} (p_i z_0^k)^q \sum_{n=q}^{N-1} z_0^{-kn} p_i^n
\]

\[
= \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k} (n \geq 1);
\] (A.4)

Equations (A.3) and (A.4) are used in (19) and (20), respectively, in the text.

**APPENDIX B**

*Derivations for the Cross Spectrum \( G_{0m}(k) \) in Section IV-B*

The expectations \( G_{0m}(k) \) of the cross spectrum between the spectrum \( X_0(k) \) of the input signal in the 0th block and the spectrum \( Z_{0m}(k) \) of the \( N \)-point signal in the \( m \)th block of the response of the transfer system to the \( x_0(n) \) in (22) and (23) is given by

\[
G_{0m}(k) = \mathbb{E}[X_0(k)Z_{0m}(k)]
\]

\[
= \mathbb{E}\left[\sum_{i=1}^{N-1} x_0^i(n)z_0^{-kn} \sum_{q=1}^{P} C_{pi} \sum_{q=1}^{N-1} (p_i z_0^k)^q \sum_{n=q}^{N-1} x_0(q)z_0^{-kn} p_i^n\right]
\]

\[
= \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} \sum_{n=q}^{N-1} \sum_{q=1}^{N-1} E[x_0(n)x_0(q)]z_0^{-kn} p_i^n
\]

\[
= \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k} (n \geq 1);
\] (B.1)

for the case of \( m \geq 1 \). Since the variance of the input signal \( x(n) \) is assumed to be \( \sigma_x^2 \) as described in Section II,

\[
E[x_0^2(n)x_0(q)] = \sigma_x^2 \delta(n - q)
\]

where

\[
\delta(n) = \begin{cases} 
  1, & \text{if } n = 0; \\
  0, & \text{otherwise}.
\end{cases}
\]

Letting each spectrum component of \( X(k) \) have the power

\[
\sigma_x^2(k) = \sigma_x^2 \frac{N}{N-k}
\] (B.2)

at the \( k \)th frequency

\[
G_{0m}(k) = \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} \sum_{n=q}^{N-1} (p_i z_0^k)^q \sum_{n=q}^{N-1} x_0(n)z_0^{-kn} p_i^n
\]

\[
= N \sigma_x^2(k) \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} \sum_{n=q}^{N-1} (p_i z_0^k)^q \sum_{n=q}^{N-1} x_0(n)z_0^{-kn} p_i^n
\]

\[
= N \sigma_x^2(k) \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k} (n \geq 1).
\] (B.3)

For the case of \( m = 0 \), using (23), the cross spectrum \( G_{00}(k) \) is given by

\[
G_{00}(k) = \mathbb{E}[X_0^2(k)Z_{00}(k)]
\]

\[
= \mathbb{E}\left[\sum_{i=1}^{N-1} x_0^i(n)z_0^{-kn} \sum_{q=1}^{P} C_{pi} \sum_{q=1}^{N-1} (p_i z_0^k)^q \sum_{n=q}^{N-1} x_0(q)z_0^{-kn} p_i^n\right]
\]

\[
= \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} \sum_{n=q}^{N-1} \sum_{q=1}^{N-1} \mathbb{E}[x_0(n)x_0(q)]z_0^{-kn} p_i^n
\]

\[
= \sum_{i=1}^{P} C_{pi} \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k} (n \geq 1).
\]

\[
= N \sigma_x^2(k) \sum_{i=1}^{P} C_{pi} \sum_{q=1}^{N-1} \frac{1 - (p_i z_0^k)^N}{1 - p_i z_0^k} (n \geq 1).
\] (B.4)

Equations (B.3) and (B.4) are used in (24) and (25), respectively, in the text.

**APPENDIX C**

*Derivations for the Transfer Function Estimates \( \hat{H}_{all}(k) \) for the Standard Cross Spectrum Method in Section IV-C*

As shown in (4), \( \hat{H}_{all}(k) \) denotes the estimate of the transfer function obtained by applying the \( MN \)-point FFT to the \( M \)-block signals \( x(n) \) and \( y(n) \), where each block consists of \( N \) points. The estimate \( \hat{H}_{all}(k) \) is rewritten by

\[
\hat{H}_{all}(k) = \frac{\mathbb{E}[X^*(k)Y(k)]}{\mathbb{E}[|X(k)|^2]}
\] (4)

where \( MN \)-point length signals \( X(k) \) and \( Y(k) \) are obtained from (13) by

\[
X(k) = \sum_{i=0}^{M-1} X_i(k) \exp(-j2\pi k_i/M),
\]

\[
Y(k) = \sum_{i=0}^{M-1} Y_i(k) \exp(-j2\pi k_i/M)
\] (C.1)
and \(X_i(k)\) and \(Y_i(k)\) are spectra obtained by applying the \(MN\)-point FFT to the zero-padded \(MN\)-point signals \(x_i(n)\) and \(y_i(n)\), respectively, as defined in (12). Since the signals \(\{x_i(n)\}\) are mutually uncorrelated, the denominator of (4) is calculated from (C.1) such as

\[
E[|X(k)|^2] = \sum_{i=0}^{M-1} E[|X_i(k)|^2]
\]

where the definition of \(E[|X(k)|^2]\) is different from that of \(E[|X(k)|^2]\) in (26).

On the other hand, the numerator of (4) is written by

\[
E[X^*(k)Y(k)] = \sum_{i=0}^{M-1} \sum_{i=0}^{M-1} E[X_i^*(k)Y_i(k)] \exp(-j2\pi \frac{k(l-i)}{M}).
\]

Using the causality of the transfer system

\[
E[X_i^*(k)Y_i(k)] = 0 \quad \text{for} \quad i > l. \quad \text{(C.4)}
\]

Since the input and output signals are assumed to be stationary, the cross spectrum \(G_{i-i}(k) = E[X_i^*(k)Y_i(k)]\) between \(X_i(k)\) and \(Y_i(k)\) in the right-hand side of (C.3) is given by

\[
G_{i-i}(k) = E[X_i^*(k)Y_i(k)] = E[X_i^*(k)Y_{i-i}(k)]. \quad \text{for} \quad l \geq i. \quad \text{(C.5)}
\]

Thus, the numerator of (4) is obtained by

\[
E[X^*(k)Y(k)] = \sum_{i=0}^{M-1} \sum_{i=0}^{M-1} G_{i-i}(k) \exp(-j2\pi \frac{k(l-i)}{M})
\]

Using this result, the term in the square brackets \([\cdot]\) in (C.9) is rewritten as follows:

\[
[C.9] = N + \frac{1}{M(1-rN)(1-rM)} \left\{ M(1-rN)^2 + rN(1-rN)(M-1) + (1-rN)r(M-1) \right\}.
\]

By substituting (C.2) and (C.3) into (4)

\[
\hat{H}_{ai}(k) = \sum_{m=0}^{M-1} \frac{M-m}{M} G_m(k) \exp(-j2\pi \frac{km}{M}). \quad \text{(C.7)}
\]
Thus, the theoretical expression of $H_{all}(k)$ is summarized as follows:

$$
\hat{H}_{all}(k) = E[X^*(k)Y(k)]
\frac{E[|X(k)|^2]}{
\sum_{i=0}^{M-1} \sum_{j=0}^{M-1} E[X^*_i(k)Y_j(k)] \exp(-j2\pi \frac{k(i-1)}{M})
} = \sum_{m=0}^{M-1} \frac{M-m}{M} \hat{H}_m(k) \exp\left(-j2\pi \frac{km}{M}\right)
= \sum_{i=1}^{M-1} C_i \frac{1}{1-p_i z_0^{-1}} \left[ 1 + \frac{1}{NM} \frac{1-(p_i z_0^{-1})^{MN}}{1-(p_i z_0^{-1})^{M-1}} \right].
$$

This expression is used in (30) of the text.

**APPENDIX D**

**Derivations of the Transfer Function Estimate $\hat{H}_{bk}(k)$ for the Proposed Method in Section IV-D**

In this Appendix, the theoretical derivation for the estimate $\hat{H}_{bk}(k)$ of the total transfer system in (14) is derived using the definition of the impulse response in (15) as follows:

By substituting the estimates $\hat{H}(k)$ of (27) into (14) and using the definition of $z_0$ in (16), the estimate $\hat{H}_{bk}(k)$ is obtained by

$$
\hat{H}_{bk}(k) = \frac{1}{N^2 \sigma_2^2} \left\{ G_{00}(k) + \sum_{m=1}^{M-1} G_{0m}(k) z_0^{-kmN} \right\}.
$$

Substituting the cross spectra $G_{00}(k)$ of (24) and $G_{0m}(k)$ of (25) into this equation

$$
\hat{H}_{bk}(k) = \sum_{i=1}^{N} C_i \frac{1}{1-p_i z_0^{-1}} \left[ 1 + \frac{1}{N} \frac{1-(p_i z_0^{-1})^{MN}}{1-(p_i z_0^{-1})^{M-1}} \right] + \frac{1}{N} \sum_{m=1}^{M-1} C_i \frac{(1-p_i z_0^{-1})(1-(p_i z_0^{-1})^{M-1})}{1-(p_i z_0^{-1})^{M}} \sum_{i=1}^{P} G_{ip} p_i^{MN} (1-(p_i z_0^{-1})^{-1})^{-1} \left[ 1 + \frac{1}{N} \frac{1-(p_i z_0^{-1})^{MN}}{1-(p_i z_0^{-1})^{M-1}} \right]
\times \left[ 1-(p_i z_0^{-1})^{-1} \right] + \frac{1}{N} \sum_{m=1}^{M-1} \frac{(1-(p_i z_0^{-1})^{-1})}{1-(p_i z_0^{-1})^{M-1}} \sum_{i=1}^{P} G_{ip} p_i^{MN} (1-(p_i z_0^{-1})^{M-1})^{-1} \sum_{i=1}^{P} G_{ip} p_i^{MN} (1-(p_i z_0^{-1})^{MN}) \right].
$$

Since the last term $\sum_{m=1}^{M-1} p_i z_0^{-1} \sum_{m=1}^{M-1} \frac{p_i z_0^{-1}}{1-(p_i z_0^{-1})^{M-1}}$ in the last equation is equal to

$$
\sum_{m=1}^{M-1} \frac{(p_i z_0^{-1})^{MN}}{1-(p_i z_0^{-1})^{M-1}} N = \frac{(p_i z_0^{-1})^{N-1}(1-(p_i z_0^{-1})^{M-1})}{1-(p_i z_0^{-1})^{N}}.
$$

This expression is used in (32) of the text.

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