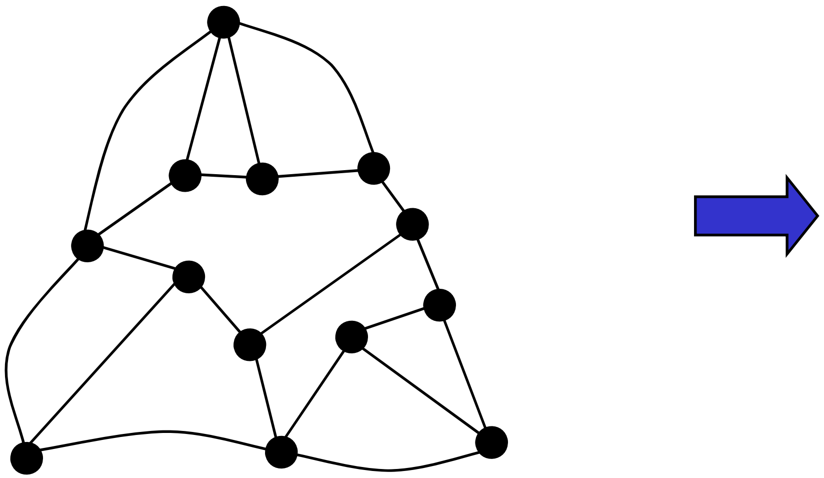
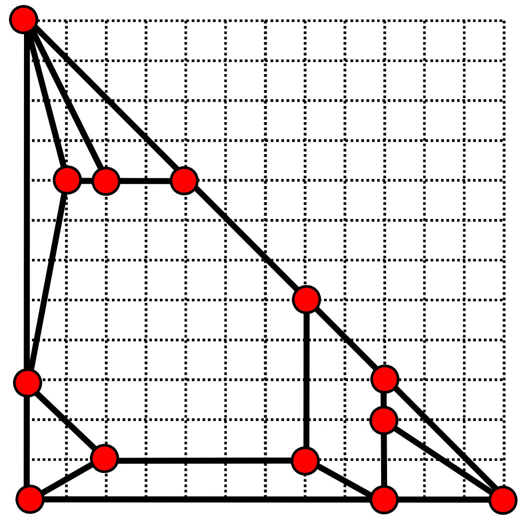
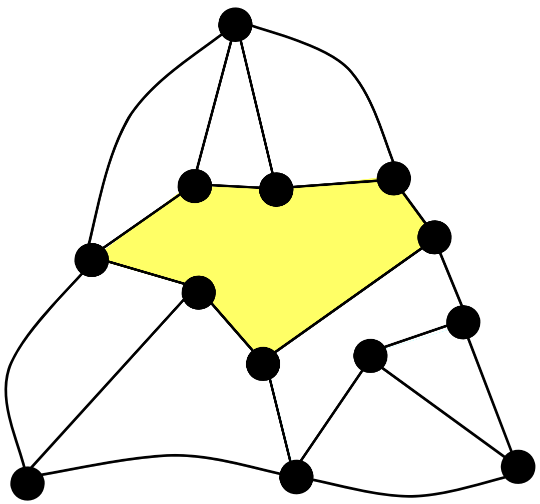
# Canonical Decomposition, Realizer, Schnyder Labeling and Orderly Spanning Trees of Plane Graphs

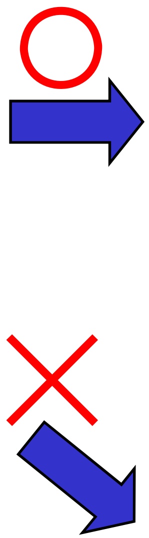


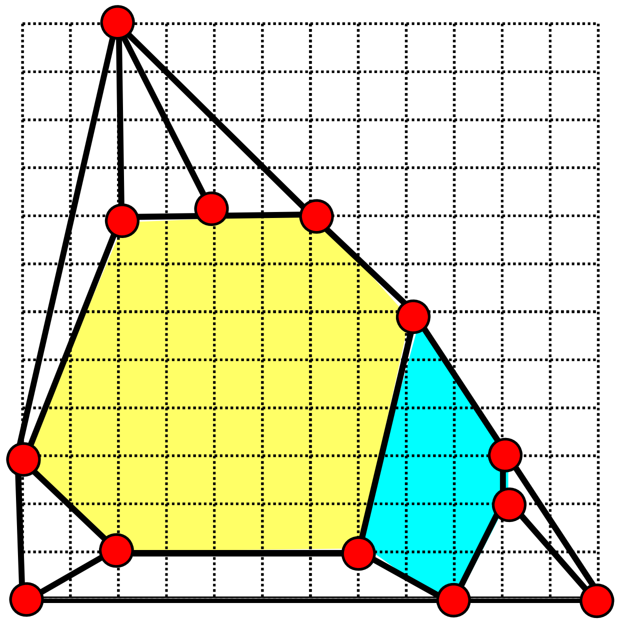


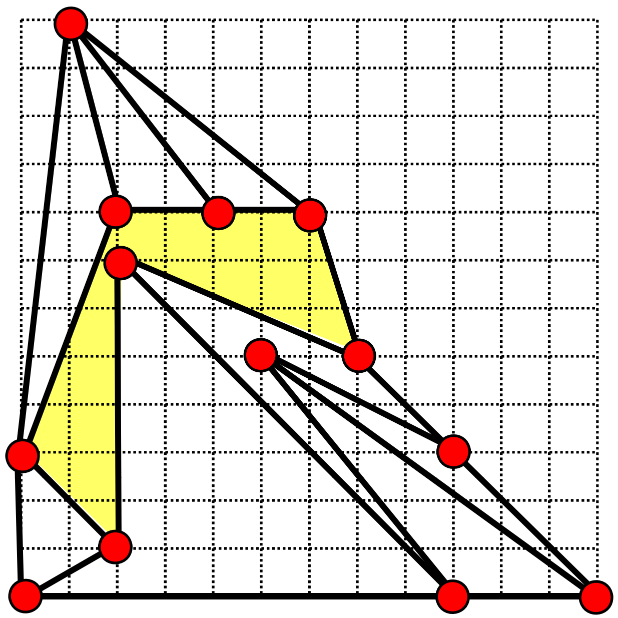
Kazuyuki Miura, Machiko Azuma and Takao Nishizeki

# Convex grid drawing



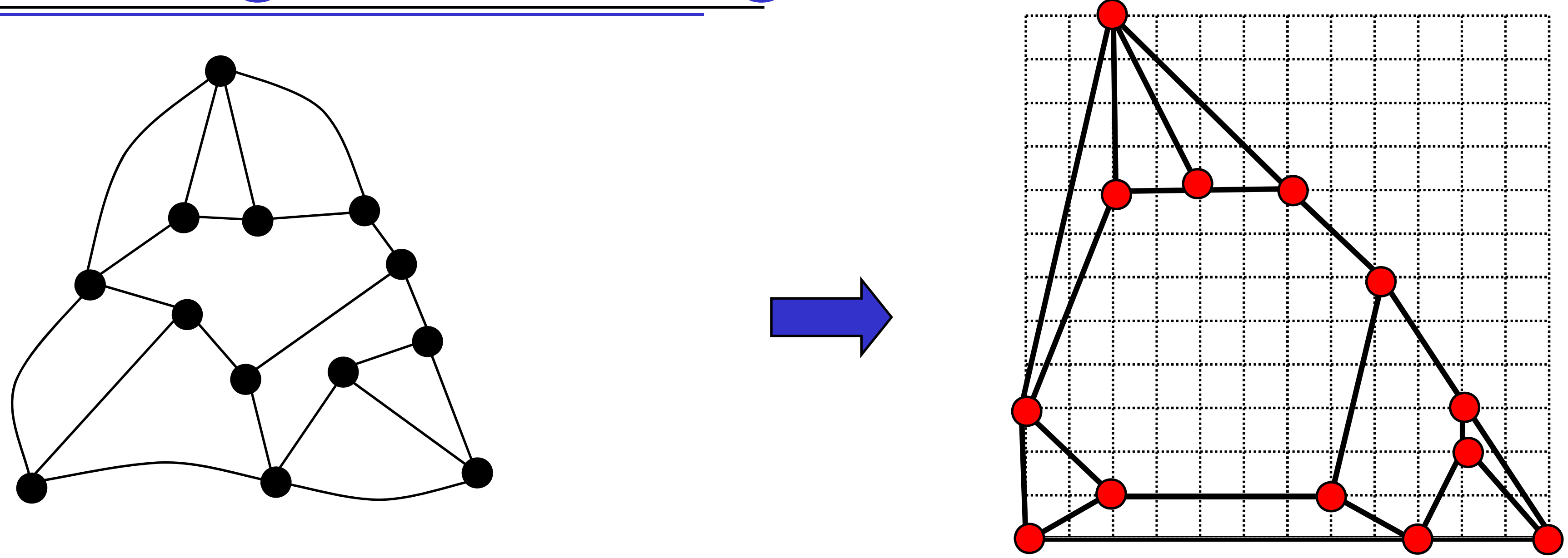






1:all vertices are put on grid points
2:all edges are drawn
as straight line segments
3:all faces are drawn
as convex polygons

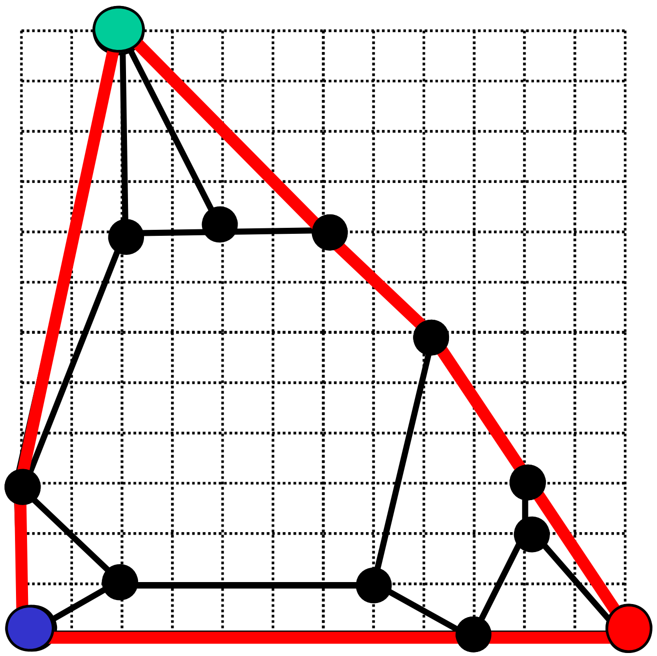
# Convex grid drawing



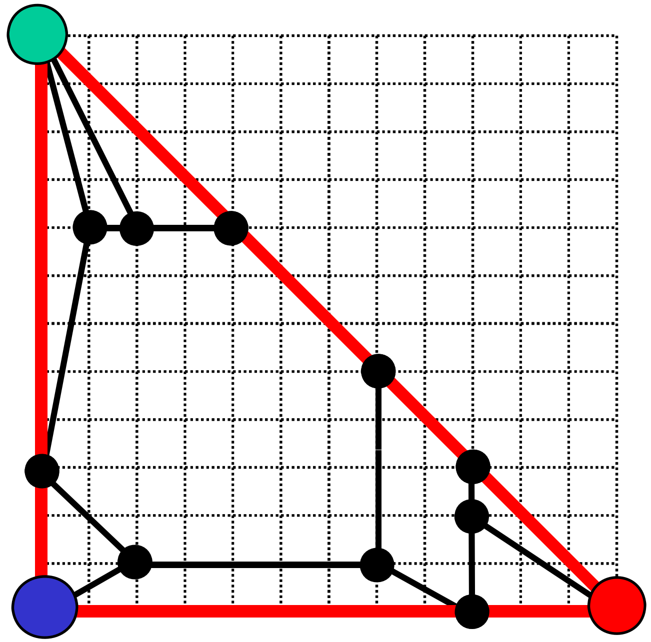
drawing methods

1. canonical decomposition
2. realizer
3. Schnyder labeling

# How to construct a convex grid drawing



[CK97]

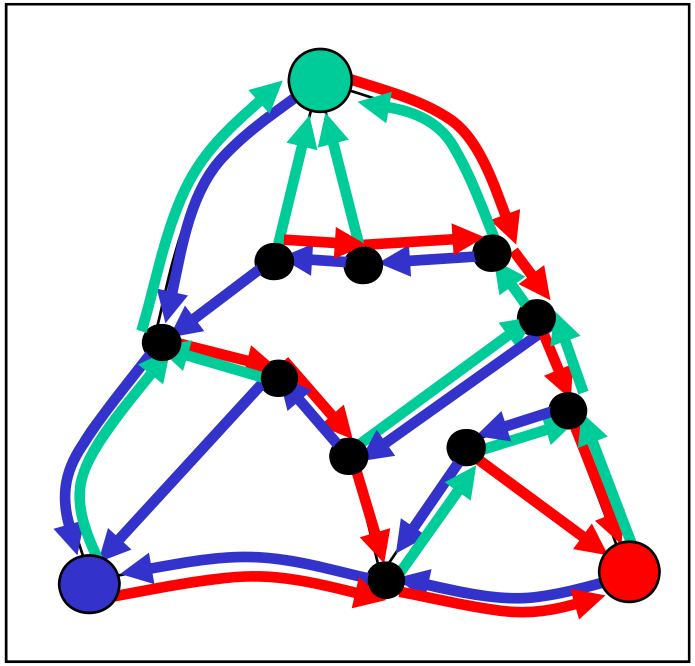


outer triangular
convex grid drawing

Ca … n [CK97]

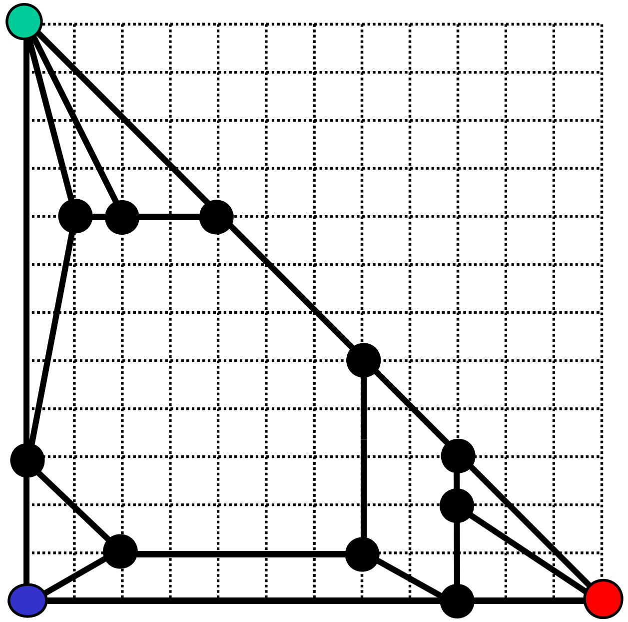
convex grid drawing
but not outer triangular
convex grid drawing

# How to construct a convex grid drawing



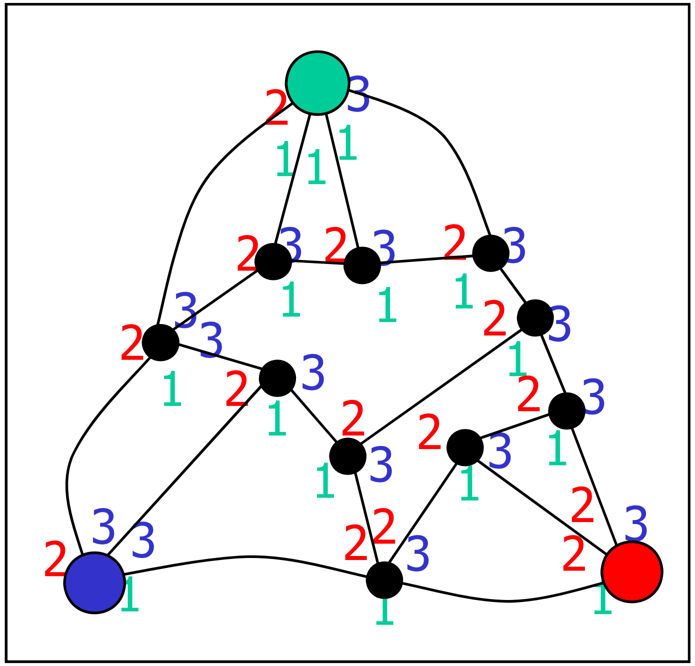
Realizer [DTV99]

[Fe01]



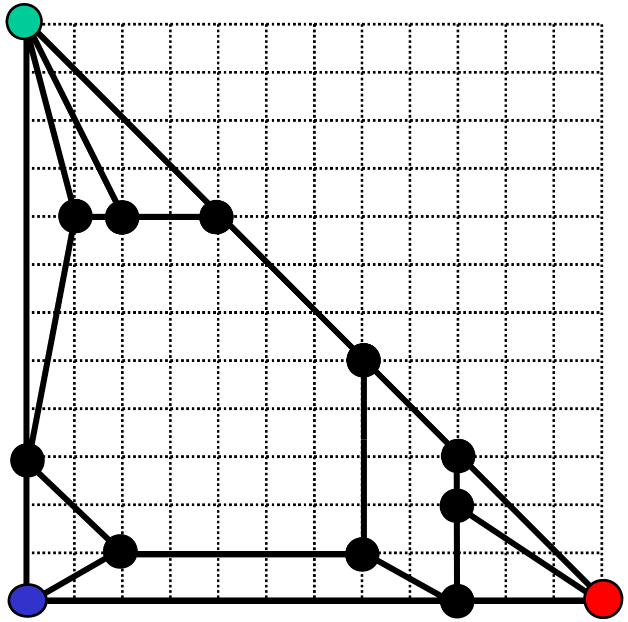
outer triangular
convex grid drawing

# How to construct a convex grid drawing



Schnyder labeling [Sc90,Fe01]

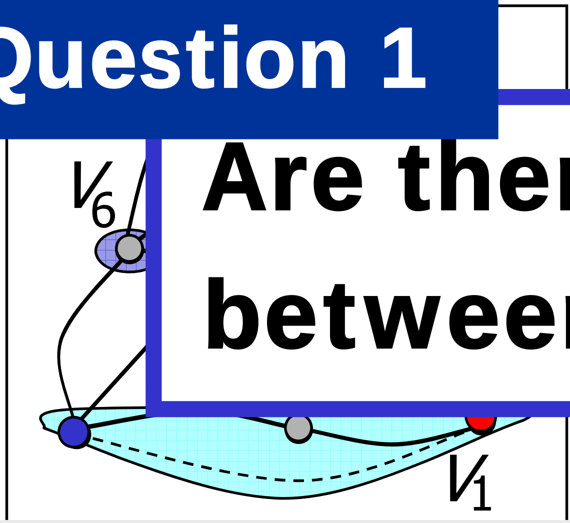
[Fe01]



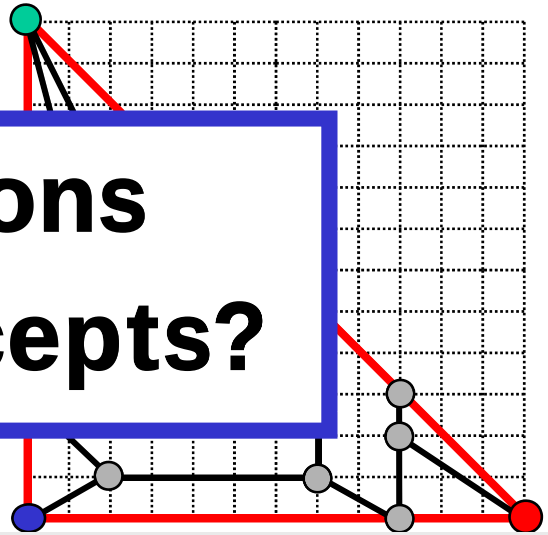
outer triangular
convex grid drawing

# Question 1

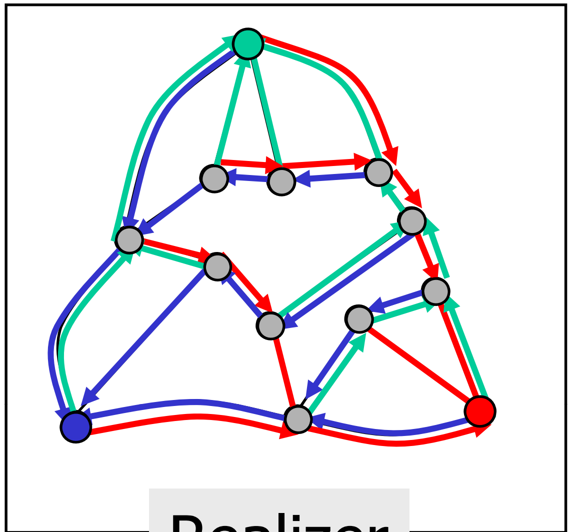
**Are there any relations between these concepts?**



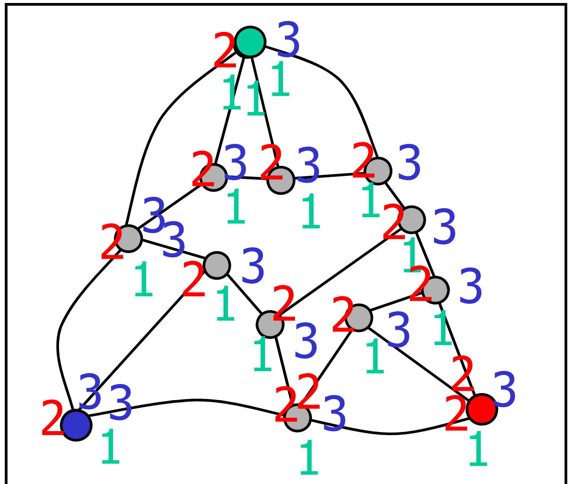

Canonical Decomposition



Convex Grid Drawing

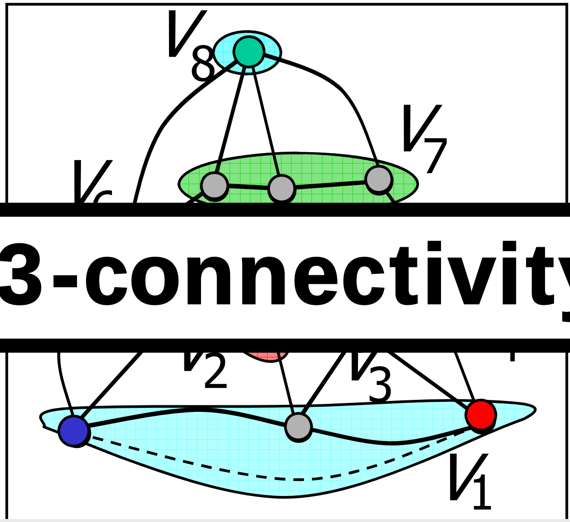


Realizer



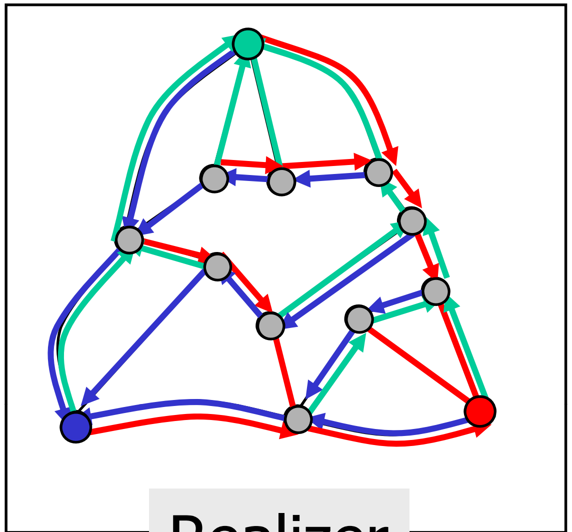
Schnyder labeling

# Known results




Canonical Decomposition

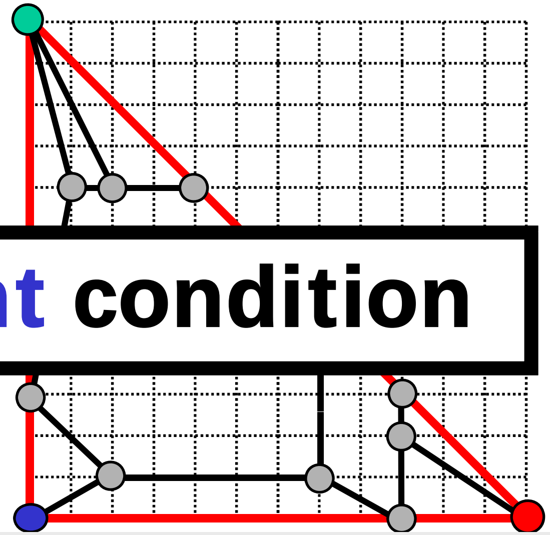
[BTV99]



Realizer

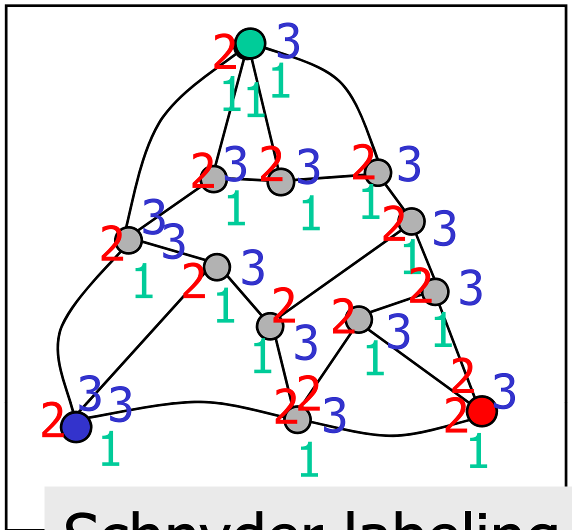
**3-connectivity is a sufficient condition**

[Fe01]



Convex Grid Drawing

[Fe01]

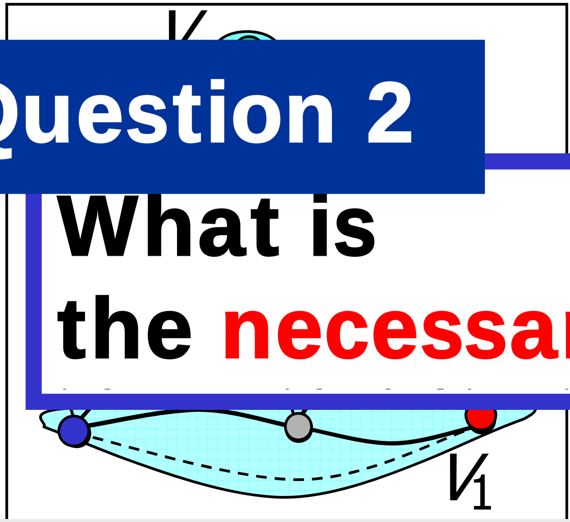


Schnyder labeling

# Known results

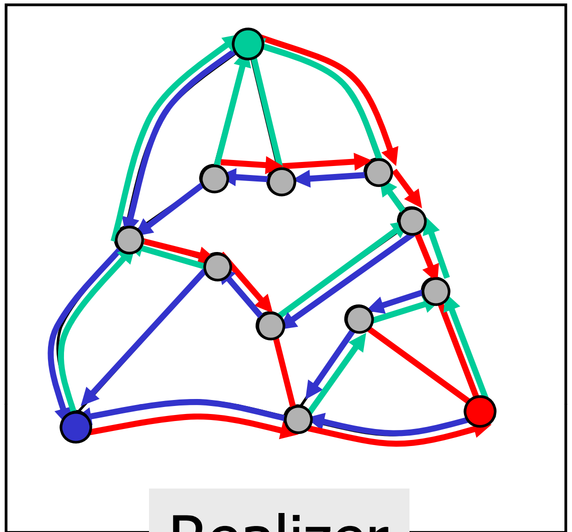
**Question 2**

**What is**
**the necessary and sufficient condition?**



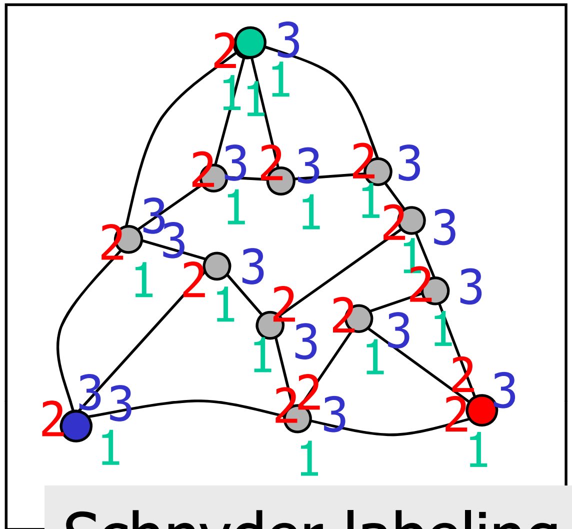

Canonical Decomposition

↓ [BTV99]



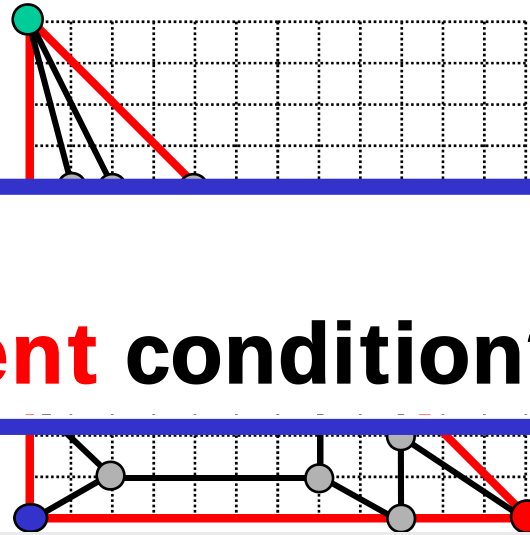
Realizer

⟺ [Fe01]



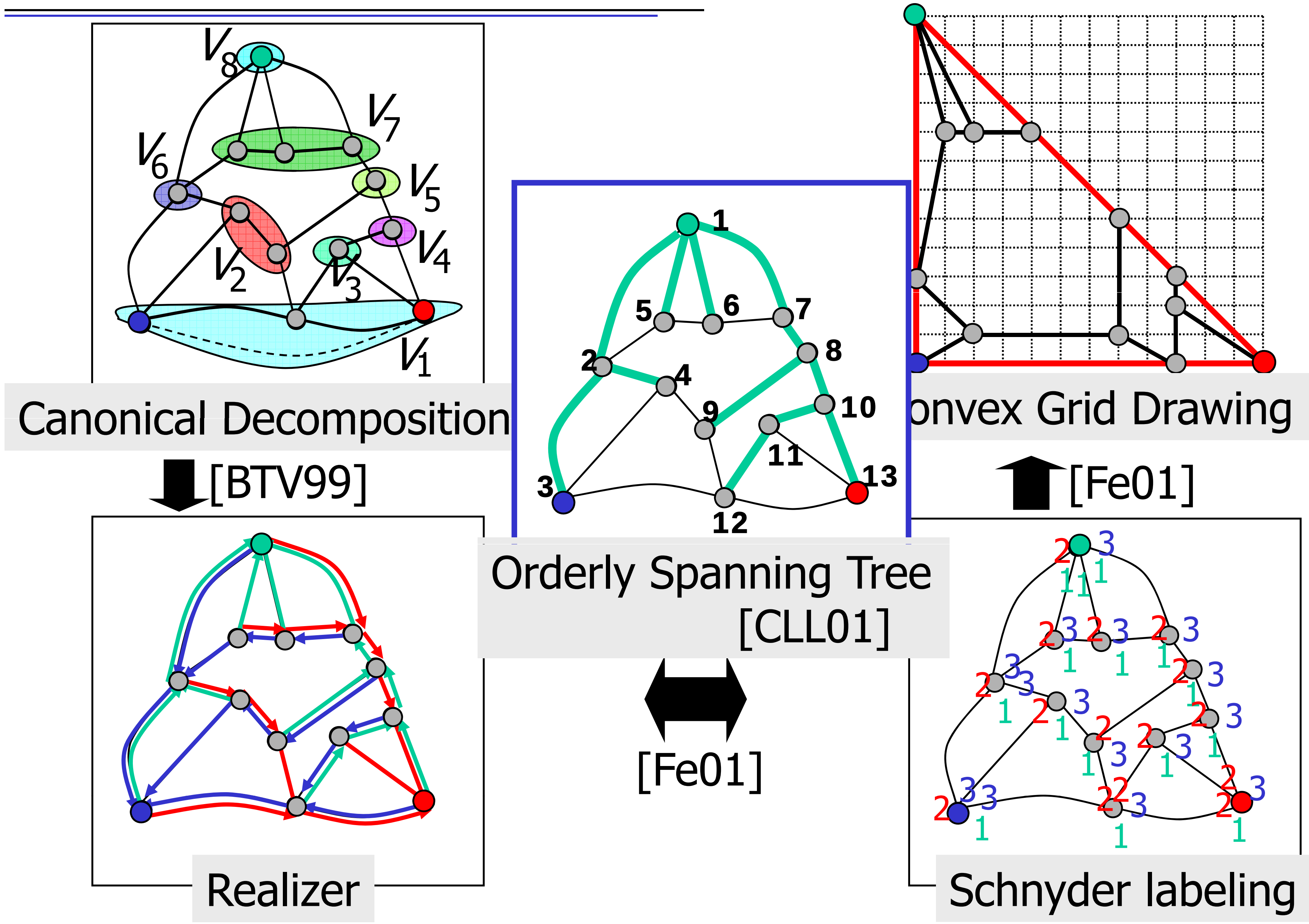
Schnyder labeling

↑ [Fe01]



Convex Grid Drawing

# Known results



# Applications of a canonical decomposition / a realizer / a Schnyder labeling / an orderly spanning tree

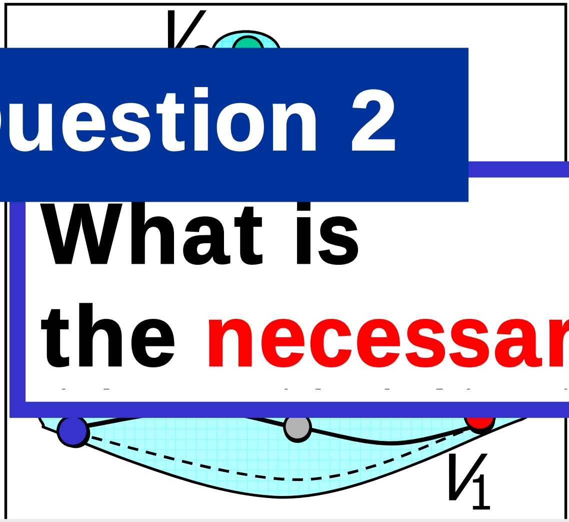
- convex grid drawing
- floor-planning
- graph encoding
- 2-visibility drawing

etc.

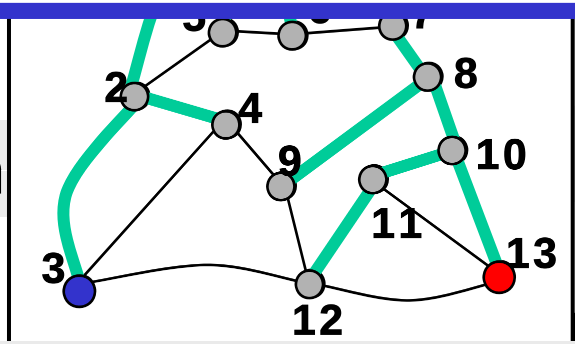
# Our results

## Question 2

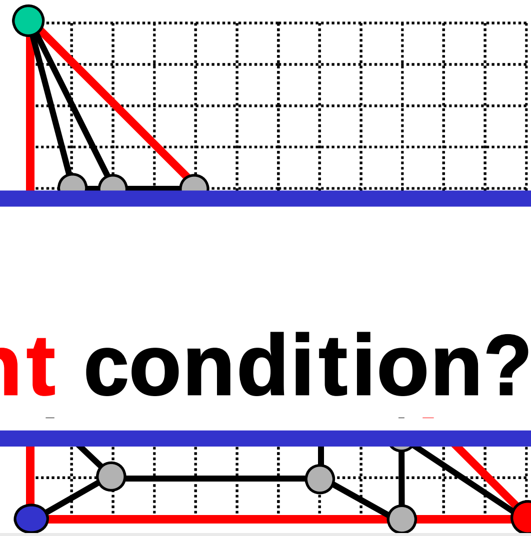
**What is the necessary and sufficient condition?**




Canonical Decomposition




Orderly Spanning Tree



Convex Grid Drawing

## Question 1

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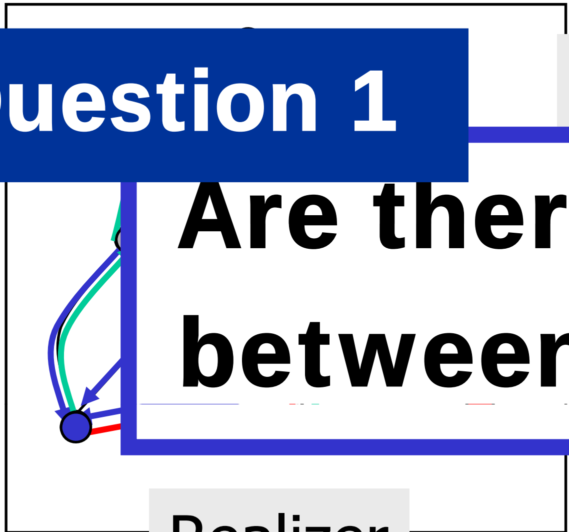
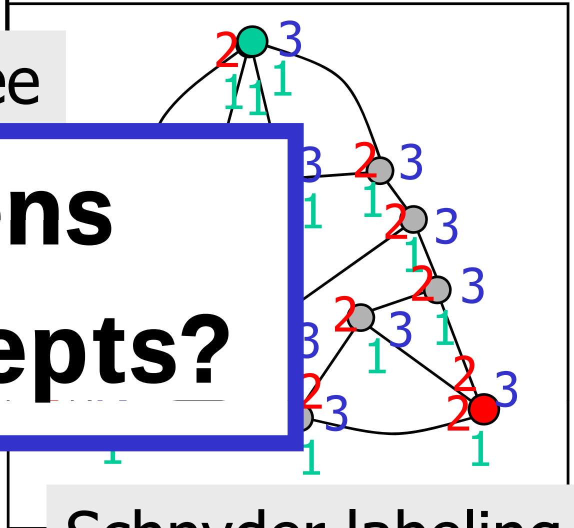
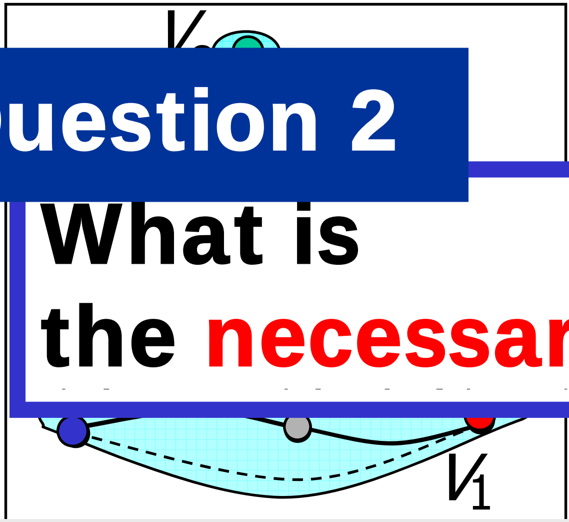


Realizer

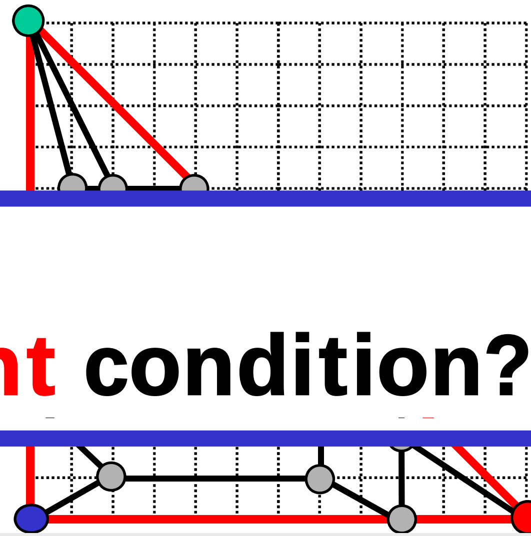


Schnyder labeling

# Our results



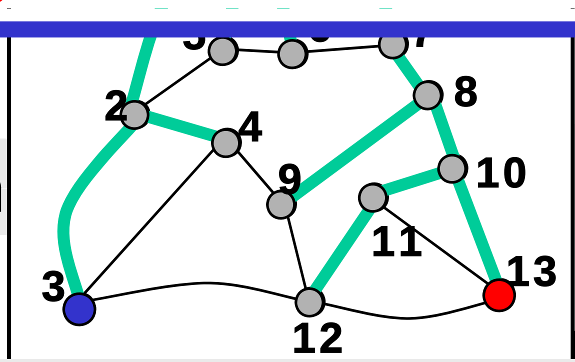
Canonical Decomposition



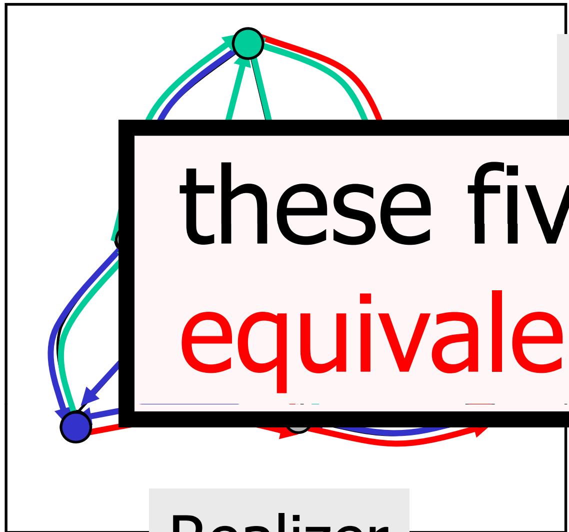
Convex Grid Drawing

**Question 2**

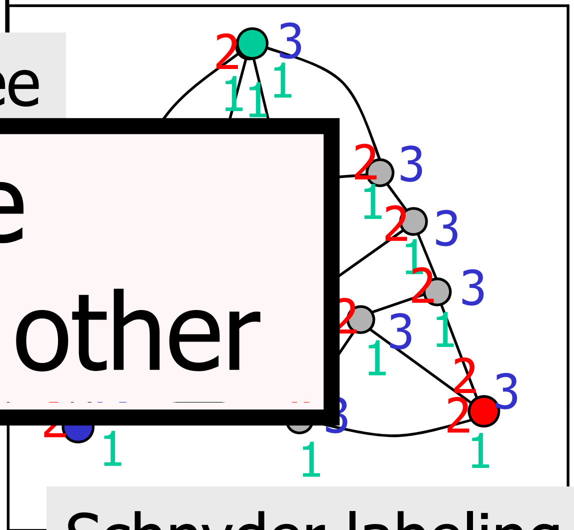
**What is the necessary and sufficient condition?**




Orderly Spanning Tree



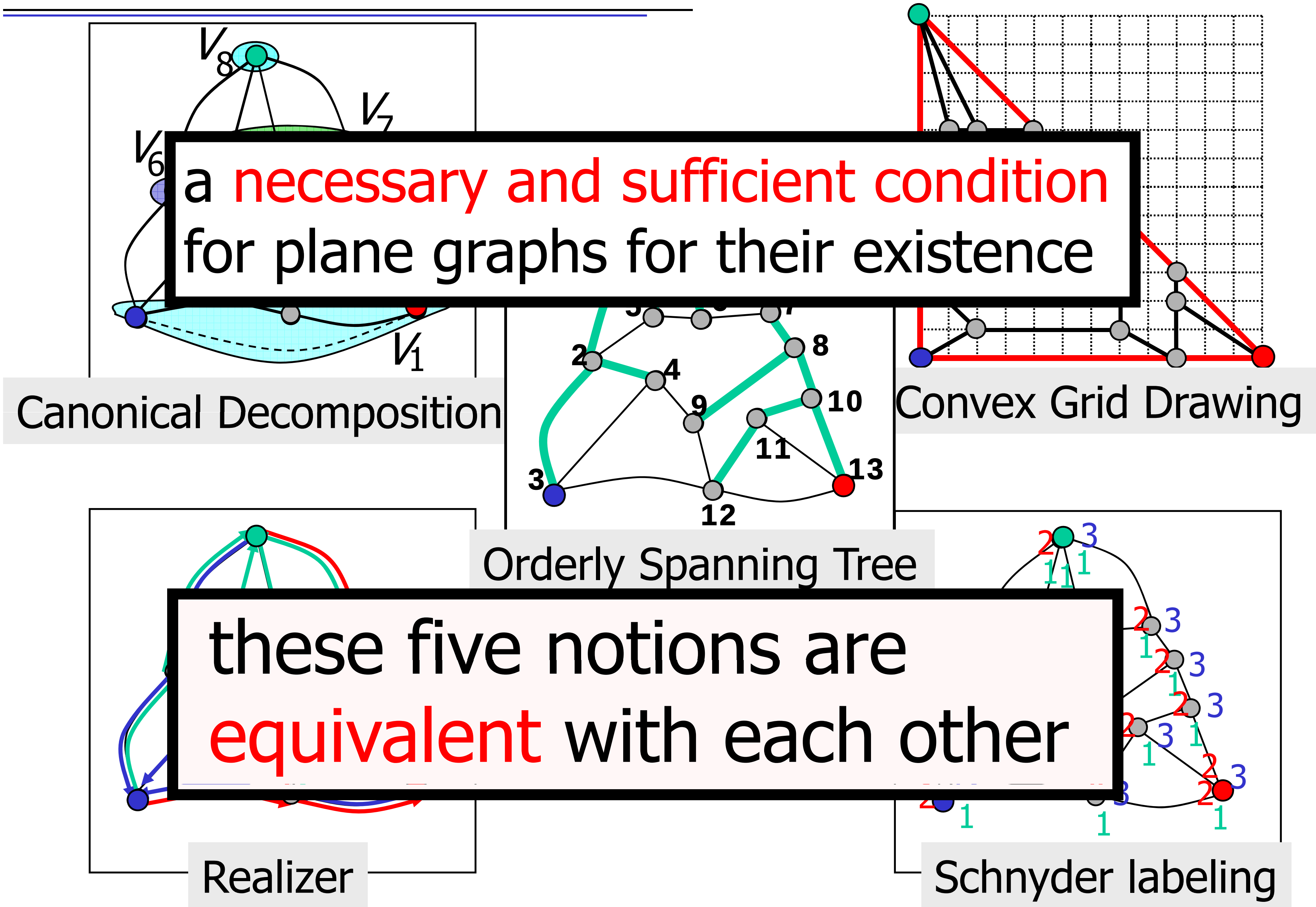
Realizer



Schnyder labeling

these five notions are equivalent with each other

# Our results




a necessary and sufficient condition for plane graphs for their existence

these five notions are equivalent with each other

## Theorem

$G$ : plane graph with each degree ≥ 3

(a) - (f) are equivalent with each other.

(a) $G$ has a canonical decomposition.

(b) $G$ has a realizer.

(c) $G$ has a Schnyder labeling.

(d) $G$ has an outer triangular convex grid drawing.

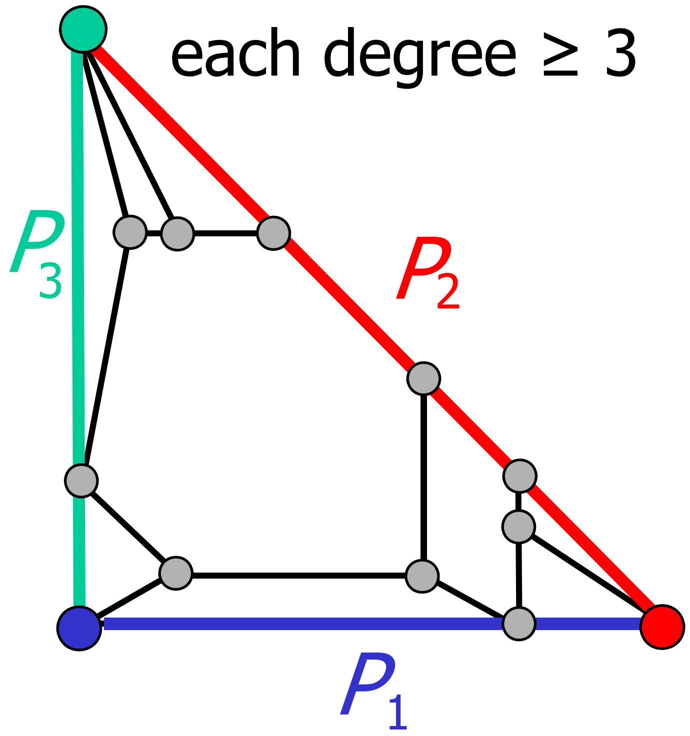
(e) $G$ has an orderly spanning tree.

(f) necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$.

# (f) Our necessary and sufficient condition

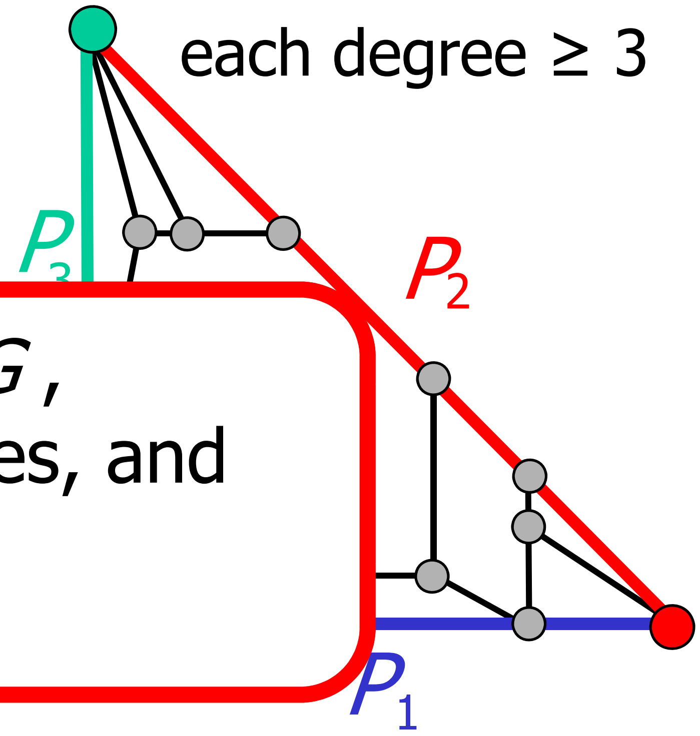
- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u,v\}$ s.t. both $u$ and $v$ are on the same path $P_i$ ($1 \le i \le 3$).

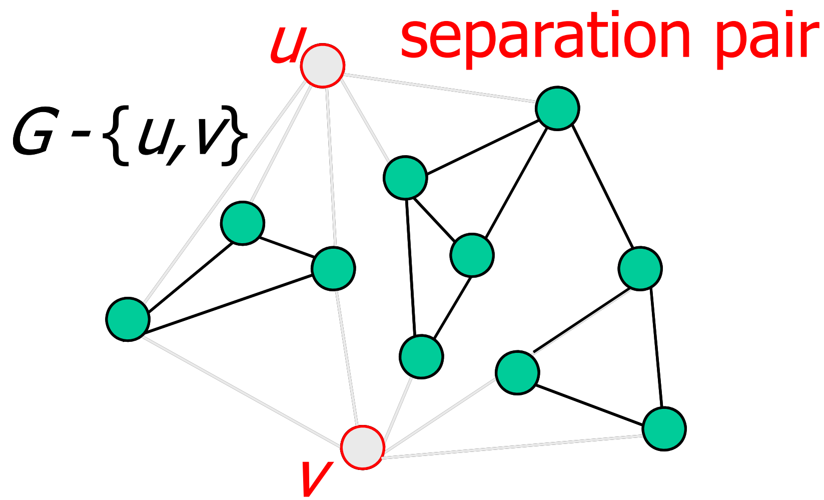


# (f) Our necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no s[...]aration pair $\{u,v\}$
  s.t. bot[...] are

For any separation pair $\{u,v\}$ of $G$,
1) both $u$ and $v$ are outer vertices, and
2) each component of $G - \{u,v\}$ contains an outer vertex.



# (f) Our necessary and sufficient condition

- *G* is internally 3-connected.
- *G* has no [se]paration pair {*u*,*v*} s.t. bot[h] ... are

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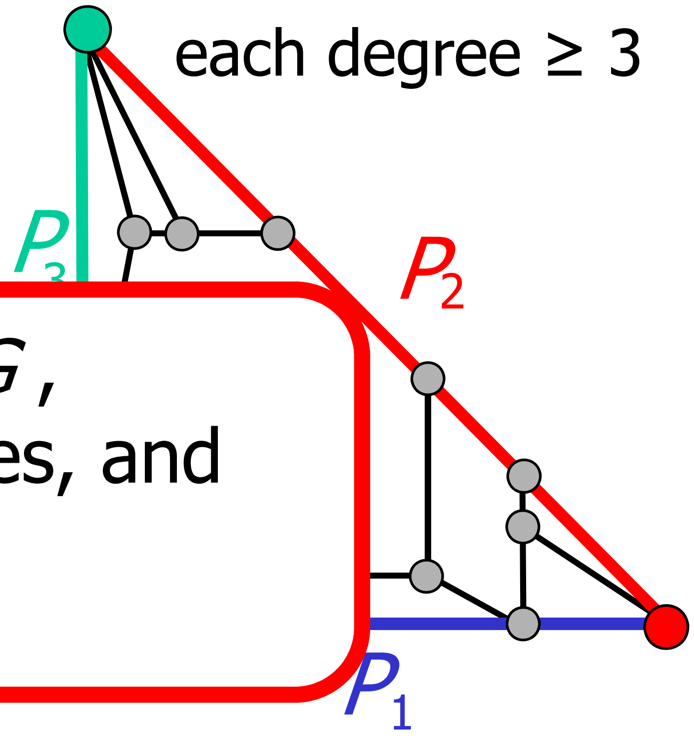


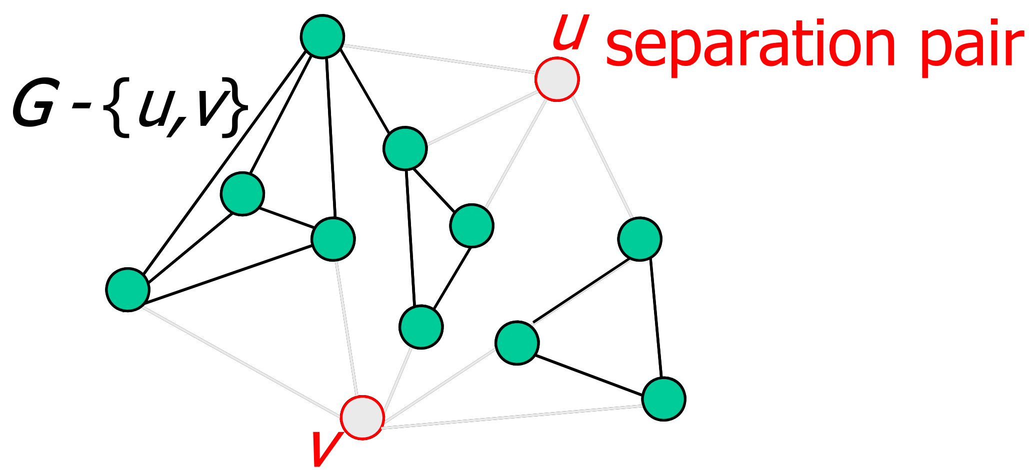

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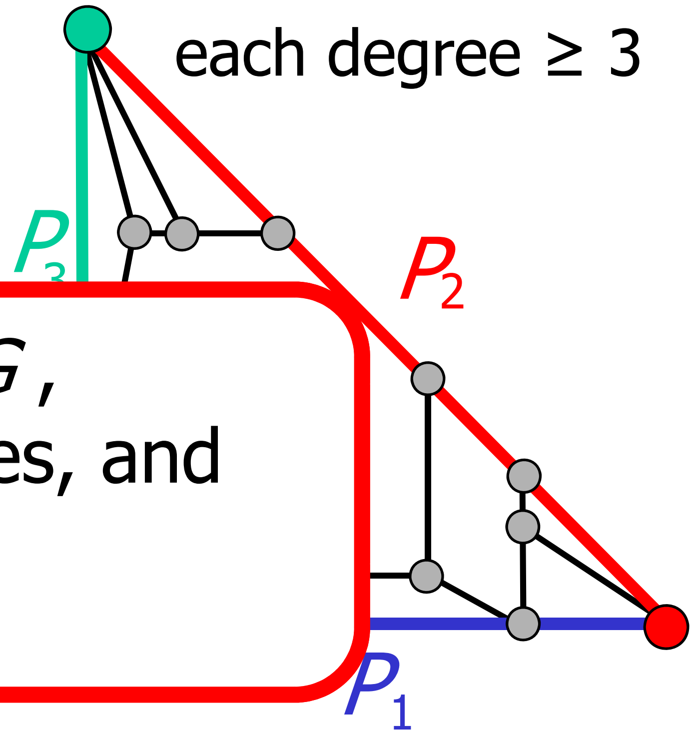
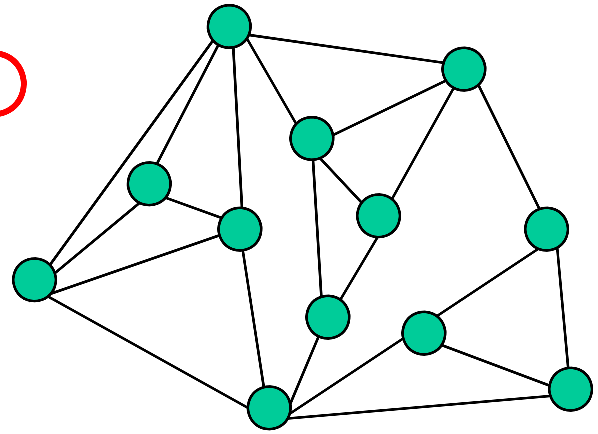



○



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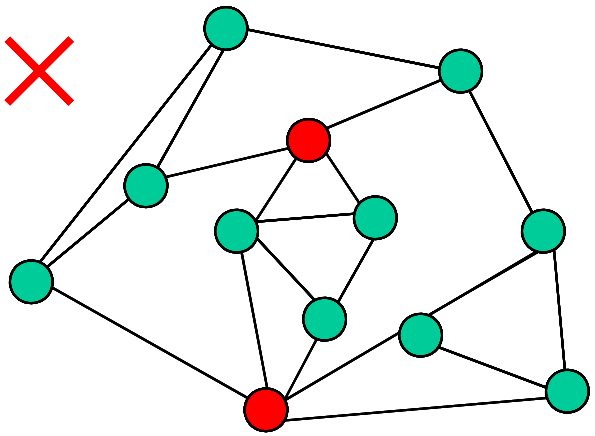
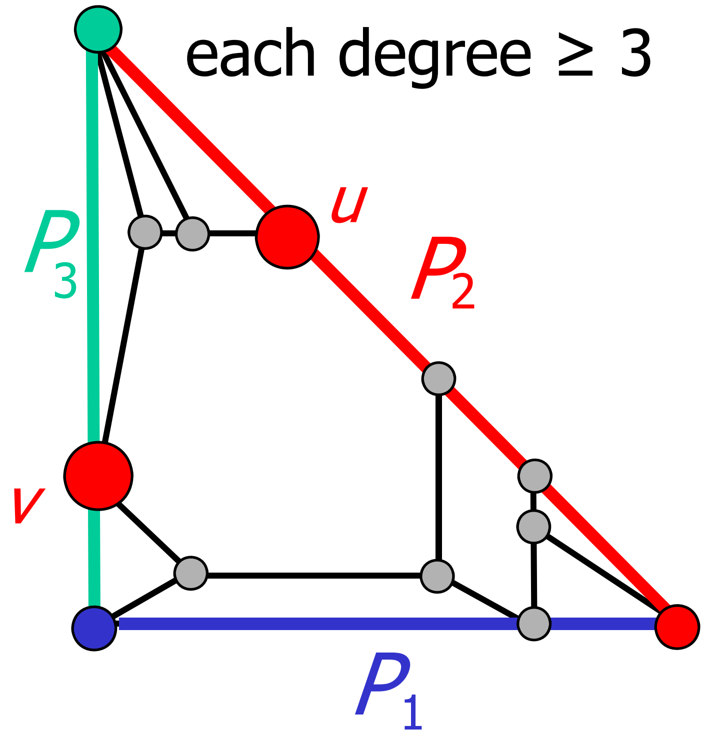


# (f) Our necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u,v\}$ s.t. both $u$ and $v$ are on the same path $P_i$ $(1 \le i \le 3)$.



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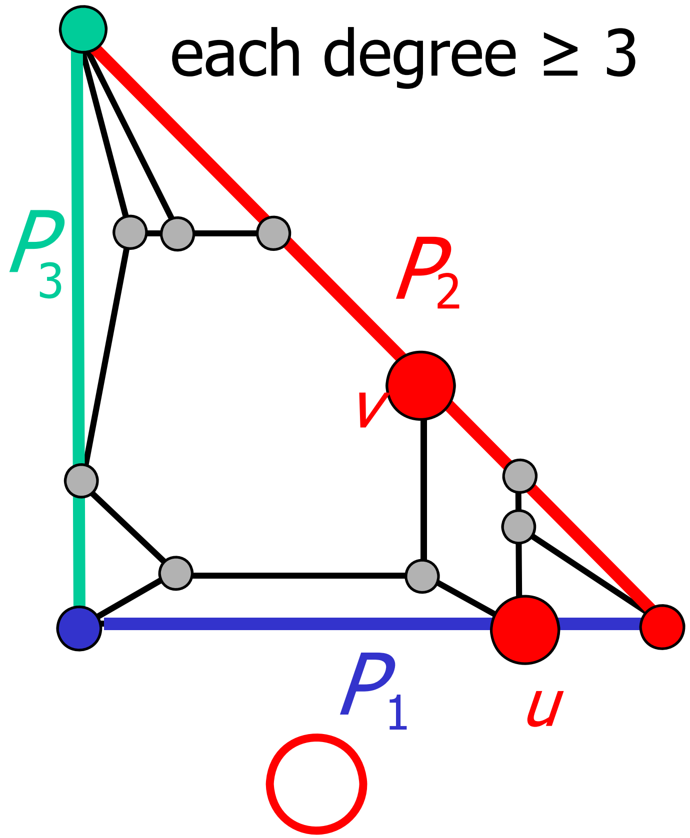

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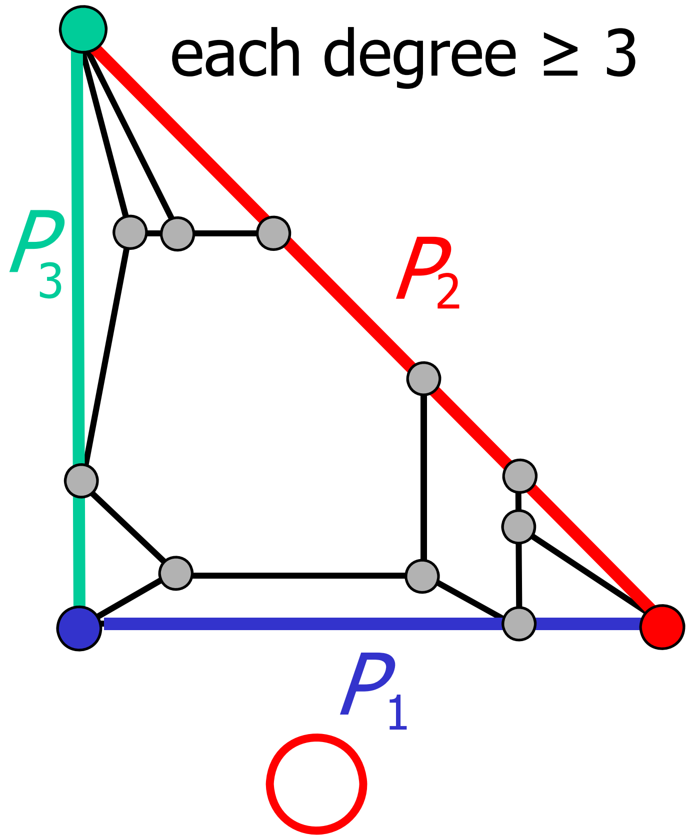
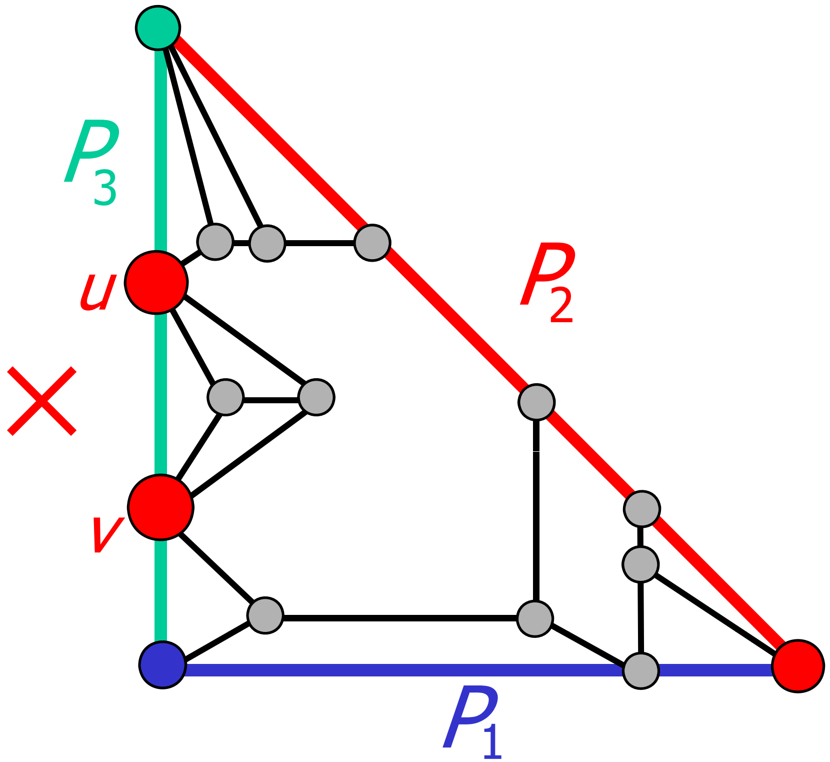


# Theorem

$G$ : plane graph with each degree $\geq 3$

(a) - (f) are equivalent with each other.

(a) $G$ has a canonical decomposition.

(b) $G$ has a realizer.

(c) $G$ has a Schnyder labeling.

(d) $G$ has an outer triangular convex grid drawing.

(e) $G$ has an orderly spanning tree.

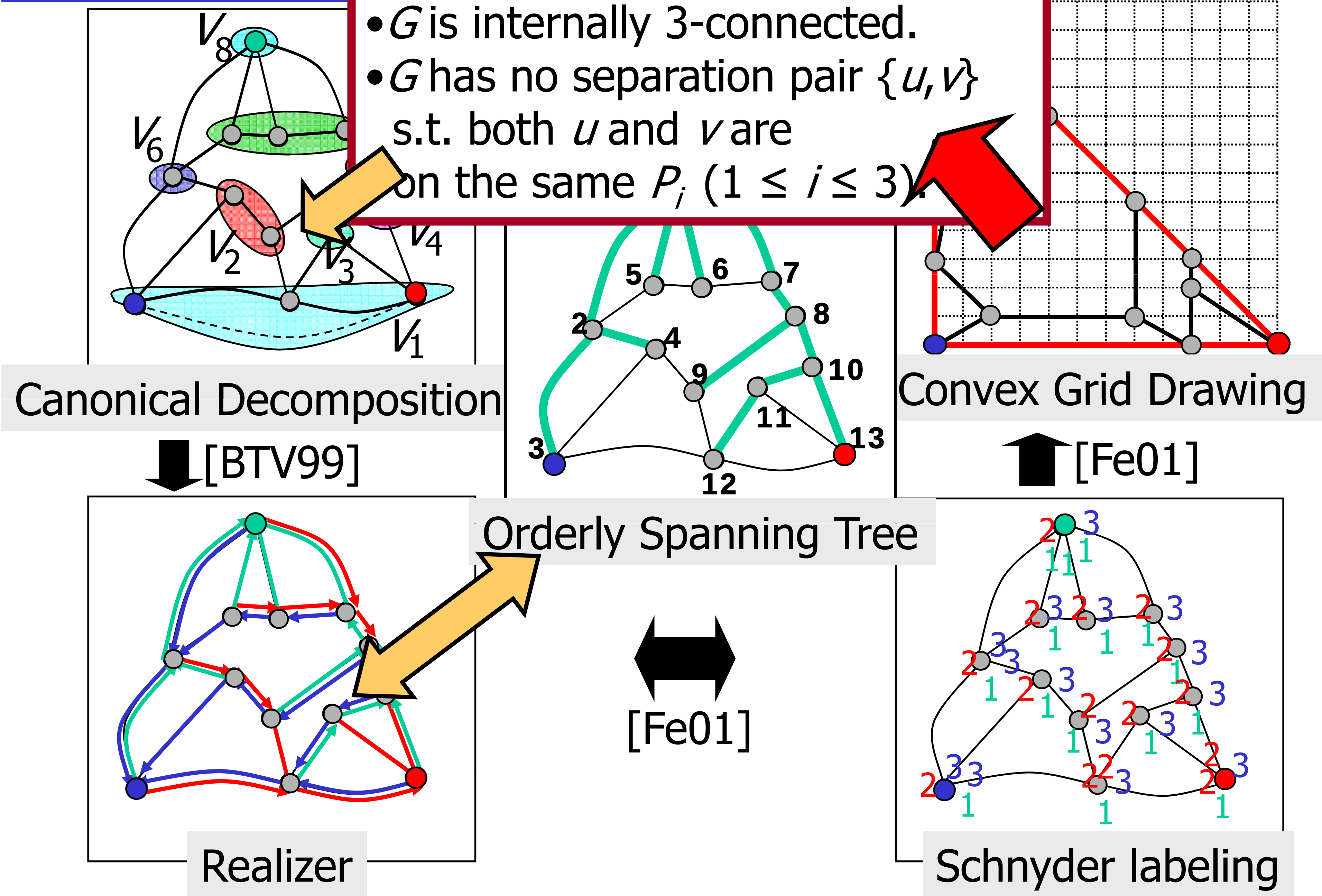
(f) necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$.

# Our results

necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u,v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$.



$G$ has an outer triangular convex grid drawing

⇒

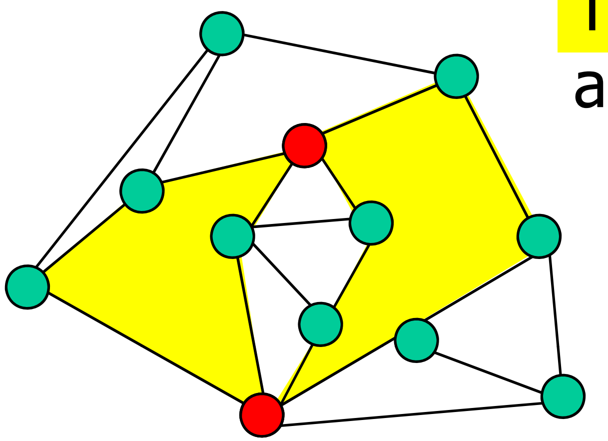
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

---

if
- $G$ is not internally 3-connected, or
- $G$ has a separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

⇒

$G$ has no outer triangular convex grid drawing



These faces cannot be simultaneously drawn as convex polygons.

**G has no convex drawing.**

*G* has an outer triangular convex grid drawing

⇒

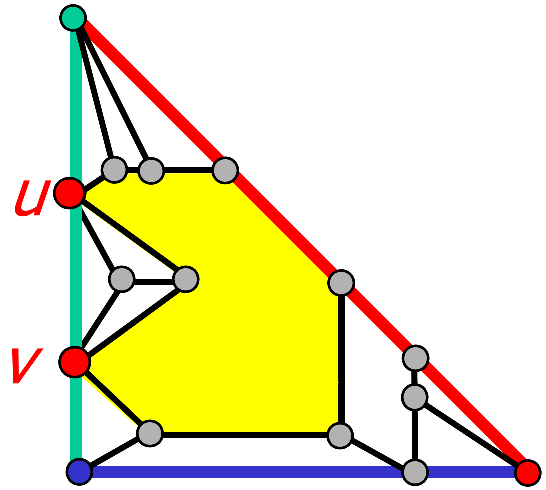
- *G* is internally 3-connected
- *G* has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$

---

if
- *G* is not internally 3-connected , or
- *G* has a separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$

⇒

*G* has no outer triangular convex grid drawing



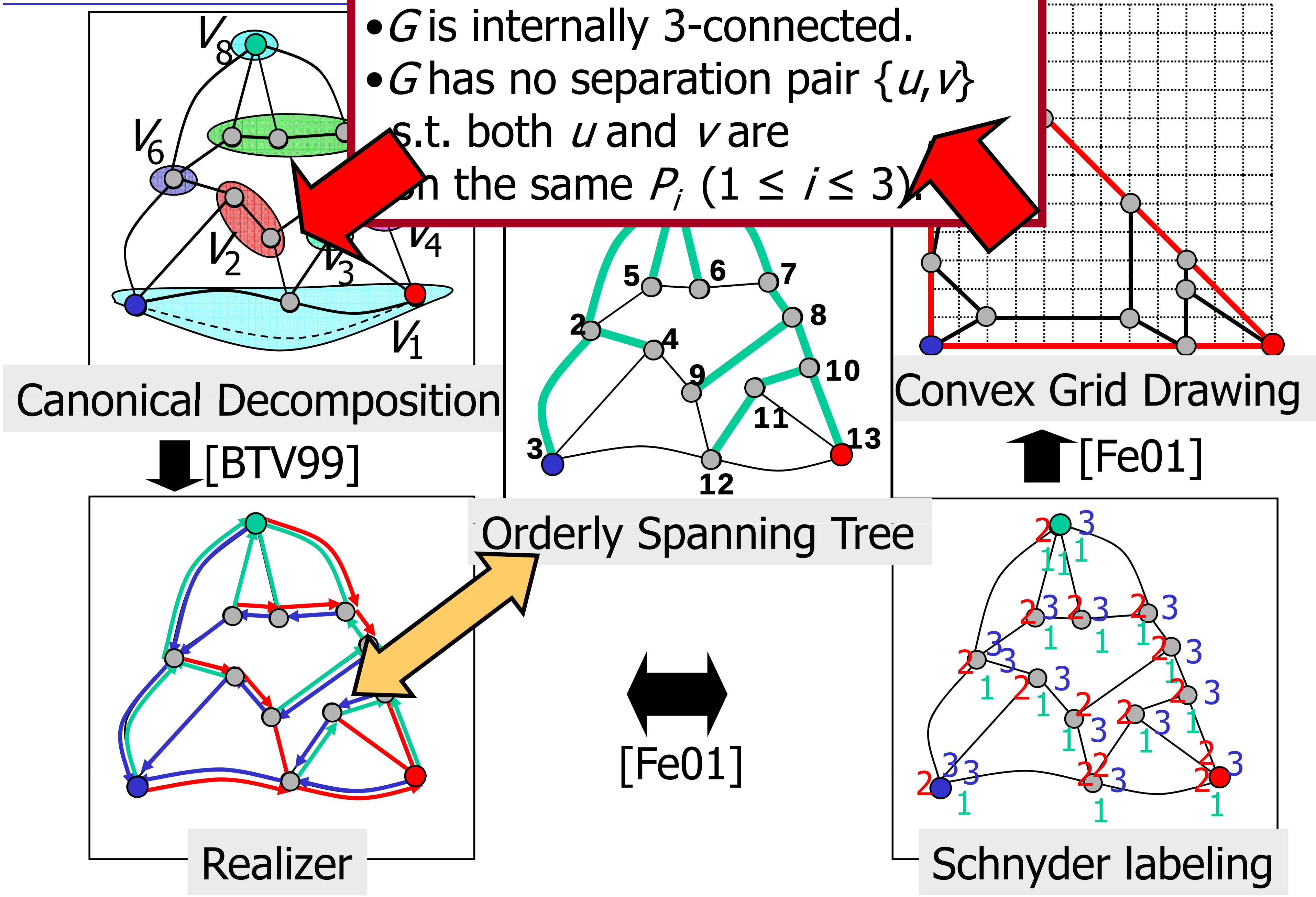

This face cannot be drawn as a convex polygon.

**G has no outer triangular convex grid drawing.**

# Our results

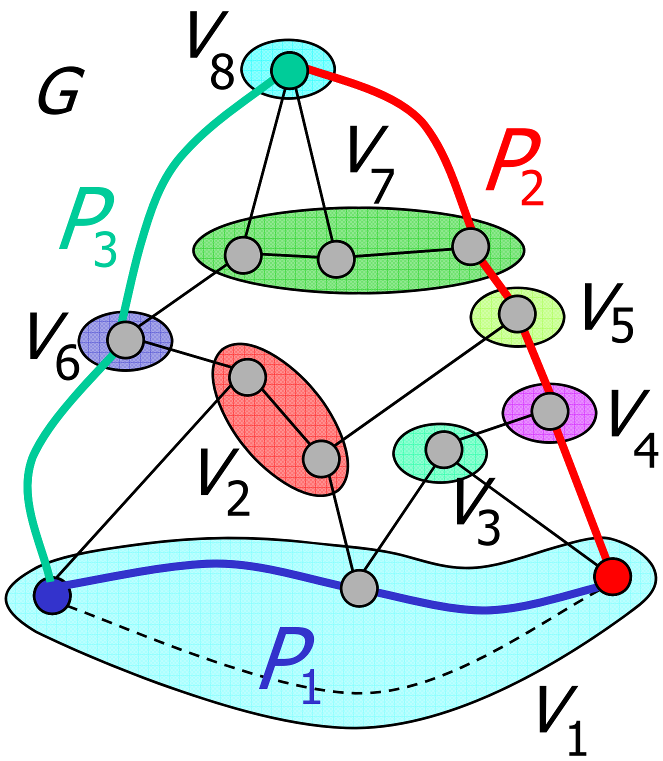
**necessary and sufficient condition**

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$.



- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$

⇒ $G$ has a canonical decomposition



- $G$ is internally 3-connected
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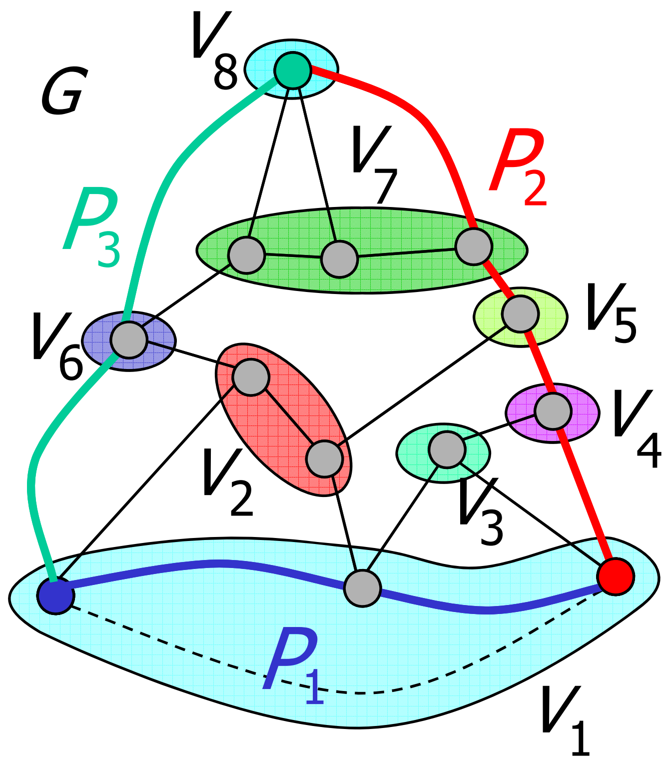
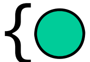
⇒ $G$ has a canonical decomposition



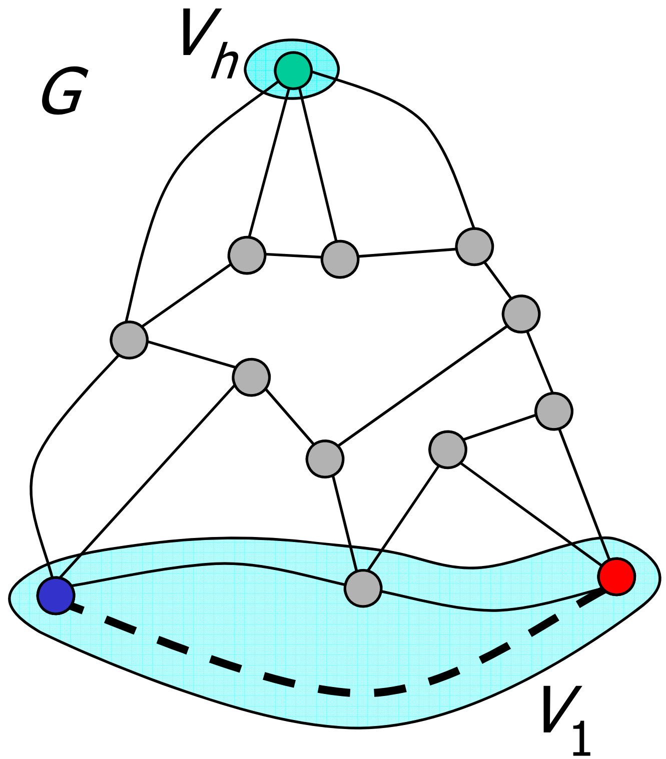

(cd1) $V_1$ consists of all vertices on the inner face containing  , and $V_h$={}.

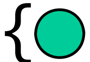
(cd2) Each $G_k$ $(1 \leq k \leq h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).

- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

⇒ $G$ has a canonical decomposition



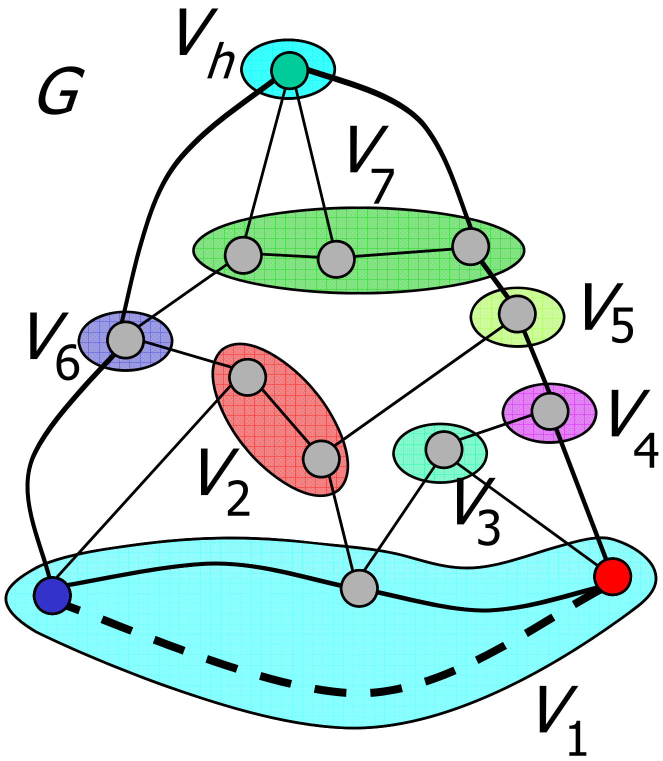

(cd1) $V_1$ consists of all vertices on the inner face containing , and $V_h$={}.

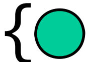
(cd2) Each $G_k$ $(1 \leq k \leq h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).

- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$

$G$ has a canonical decomposition




(cd1) $V_1$ consists of all vertices on the inner face containing  , and $V_h$={}.

(cd2) Each $G_k$ $(1 \le k \le h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \le k \le h-1)$ are outer vertices of $G_k$, and either (a) or (b).

- $G$ is internally 3-connected
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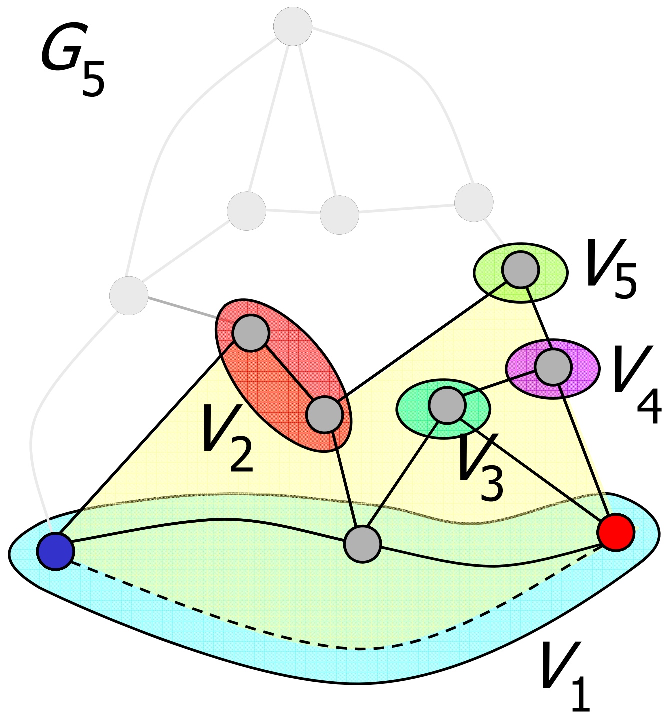
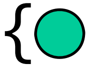
$G$ has a canonical decomposition



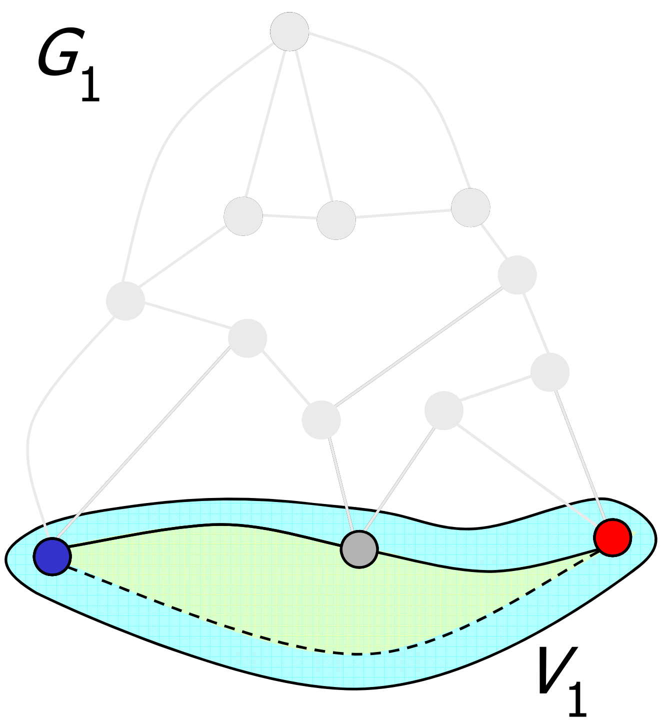

(cd1) $V_1$ consists of all vertices on the inner face containing  , and $V_h$={}.

(cd2) Each $G_k$ $(1 \le k \le h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \le k \le h-1)$ are outer vertices of $G_k$, and either (a) or (b).

- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \le i \le 3$)

⇒ $G$ has a canonical decomposition




(cd1) $V_1$ consists of all vertices on the inner face containing (blue–red edge), and $V_h$={(green vertex)}.

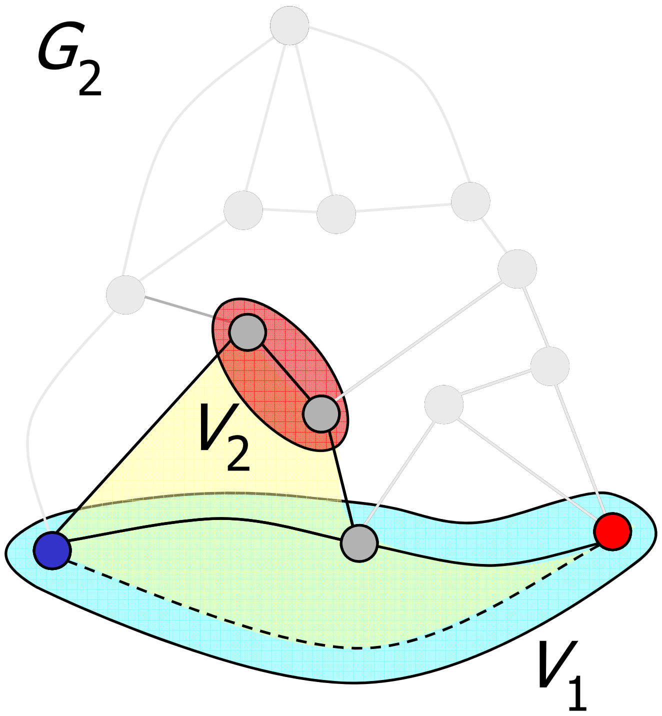
(cd2) Each $G_k$ ($1 \le k \le h$) is internally 3-connected.

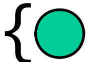
(cd3) All the vertices in each $V_k$ ($2 \le k \le h$-1) are outer vertices of $G_k$, and either (a) or (b).

- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

⇒

$G$ has a canonical decomposition




(cd1) $V_1$ consists of all vertices on the inner face containing  , and $V_h$={}.

(cd2) Each $G_k$ $(1 \leq k \leq h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h\text{-}1)$ are outer vertices of $G_k$, and either (a) or (b).

- $G$ is internally 3-connected
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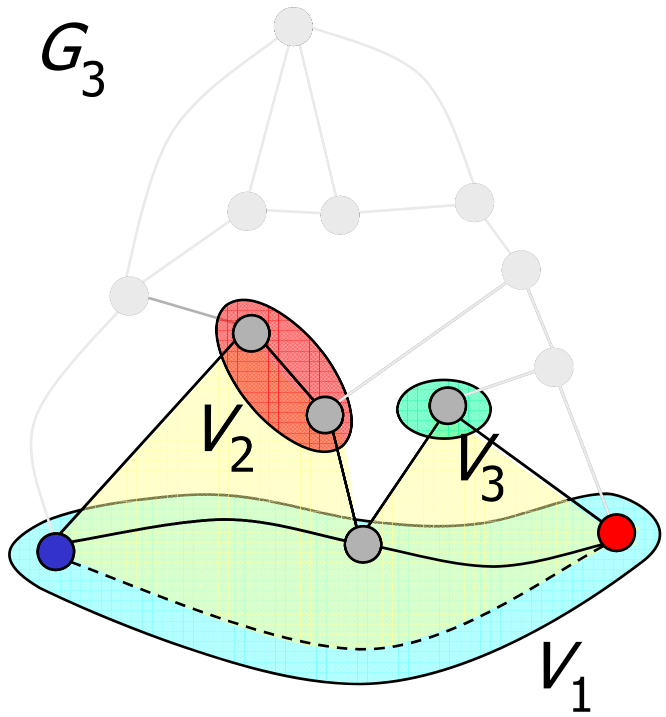
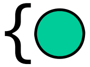
⇒ $G$ has a canonical decomposition




(cd1) $V_1$ consists of all vertices on the inner face containing  , and $V_h$={}.

(cd2) Each $G_k$ $(1 \leq k \leq h)$ is internally 3-connected.

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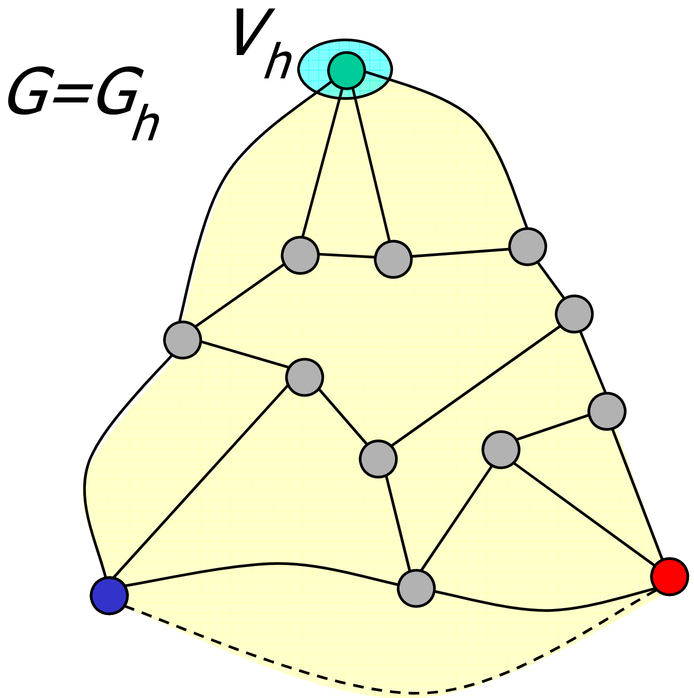
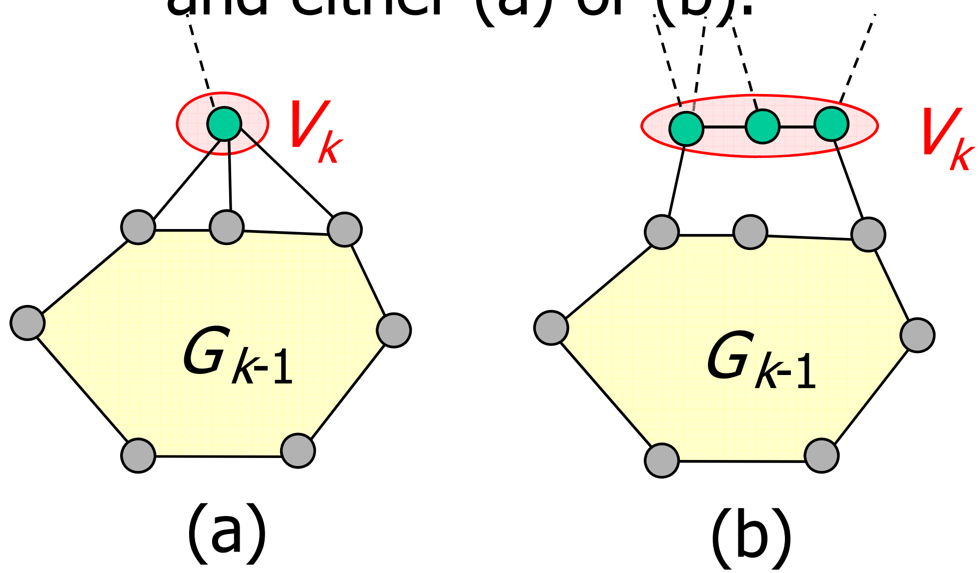
⇒ $G$ has a canonical decomposition




(cd2) Each $G_k$ $(1 \le k \le h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \le k \le h-1)$ are outer vertices of $G_k$, and either (a) or (b).

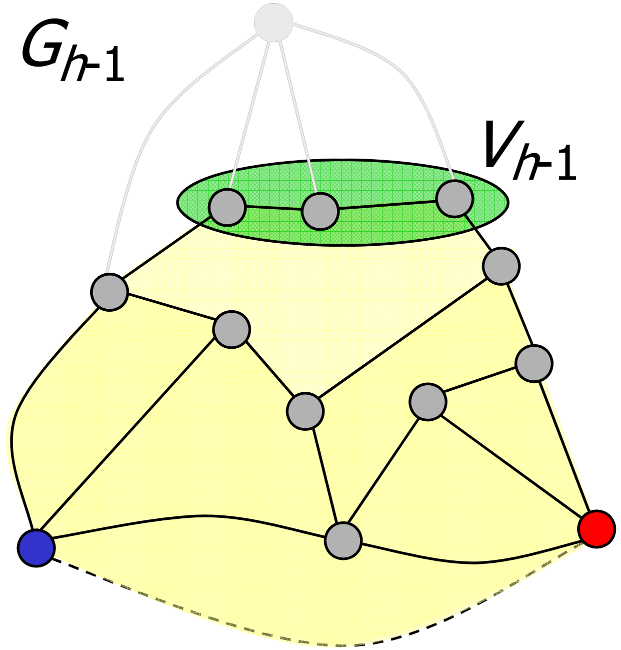


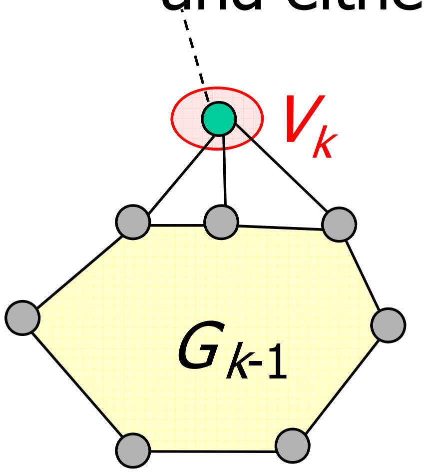
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

⇒ $G$ has a canonical decomposition

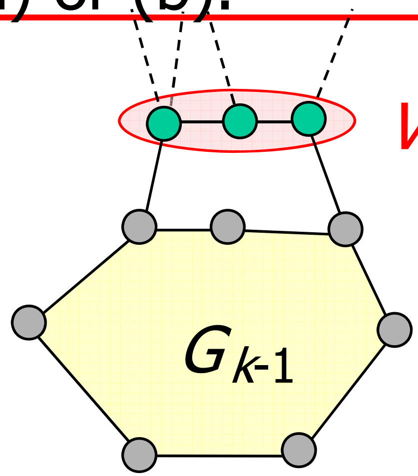
(cd2) Each $G_k$ $(1 \leq k \leq h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).



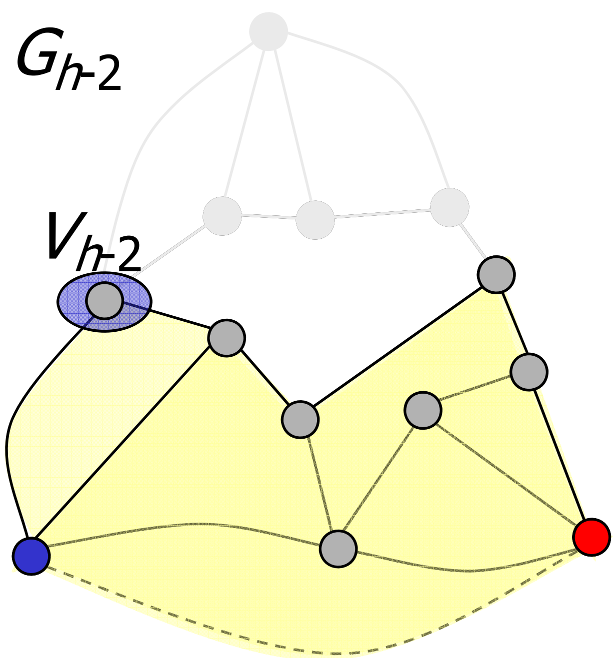

(a)




(b)

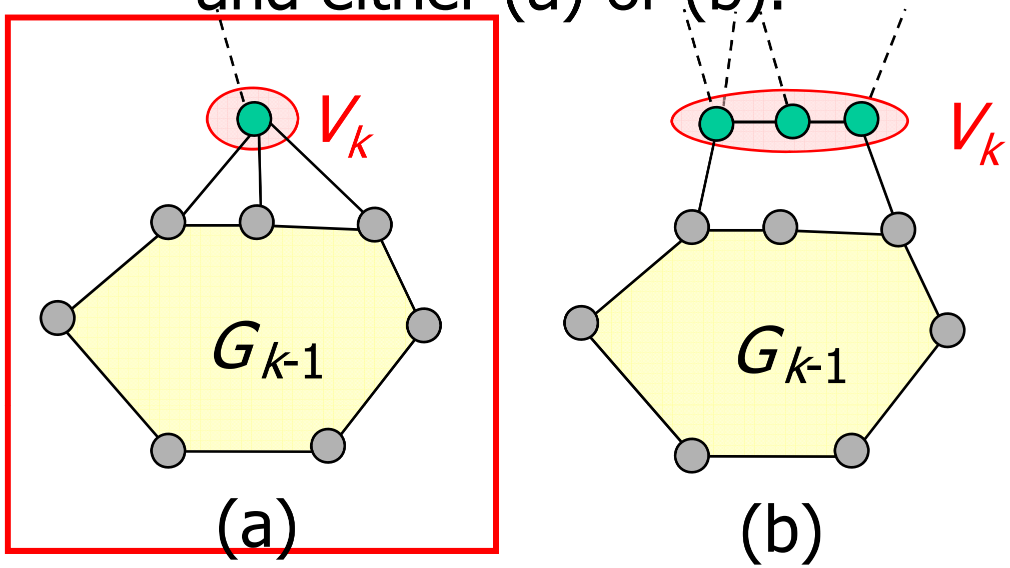
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u,v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

⇒ $G$ has a canonical decomposition



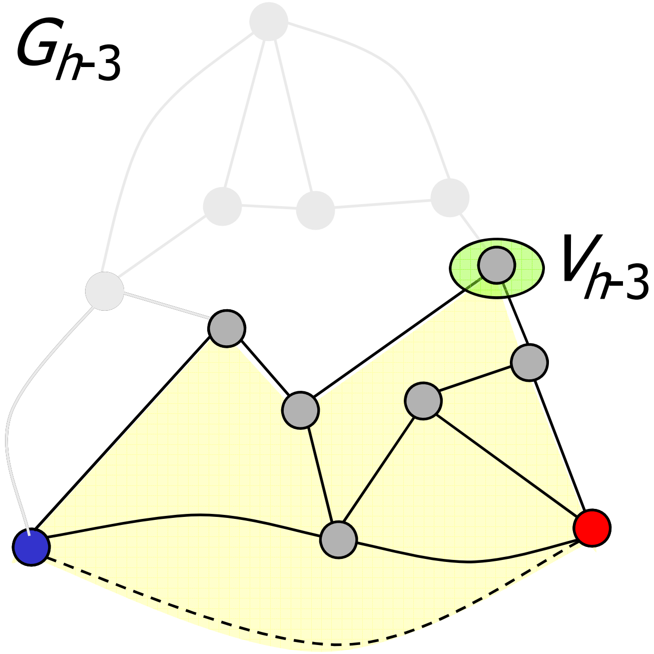

(cd2) Each $G_k$ $(1 \leq k \leq h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).



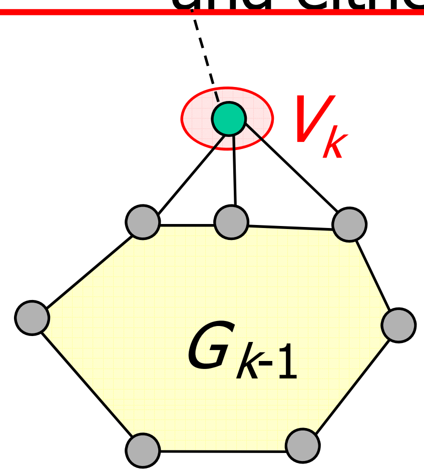
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

➡ $G$ has a canonical decomposition

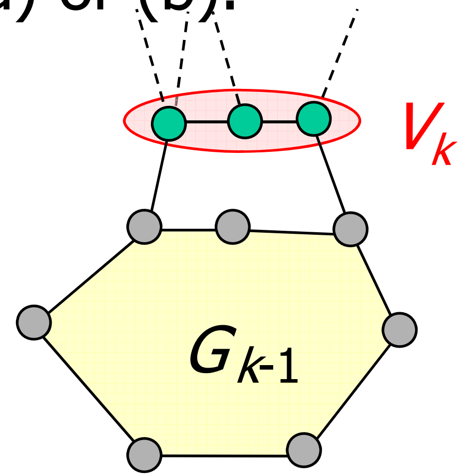



(cd2) Each $G_k$ $(1 \leq k \leq h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).



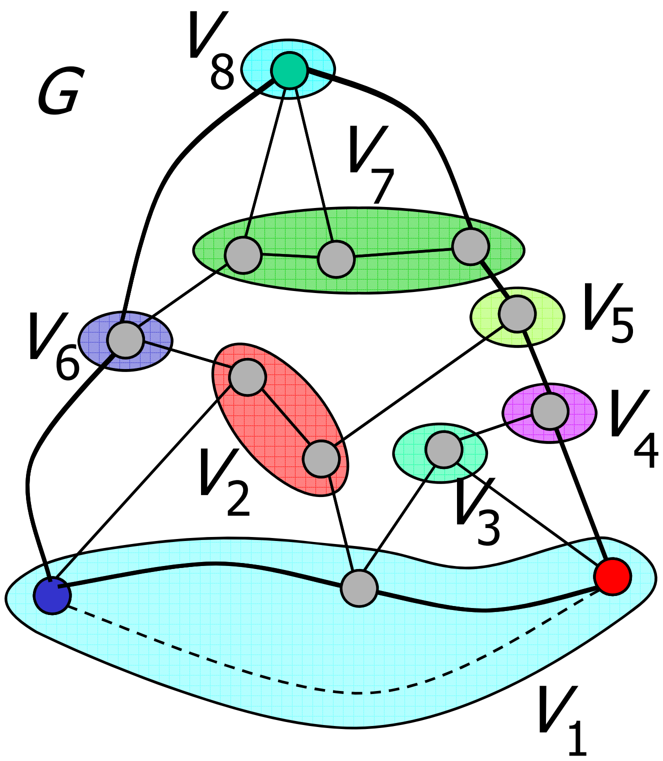

(a)




(b)

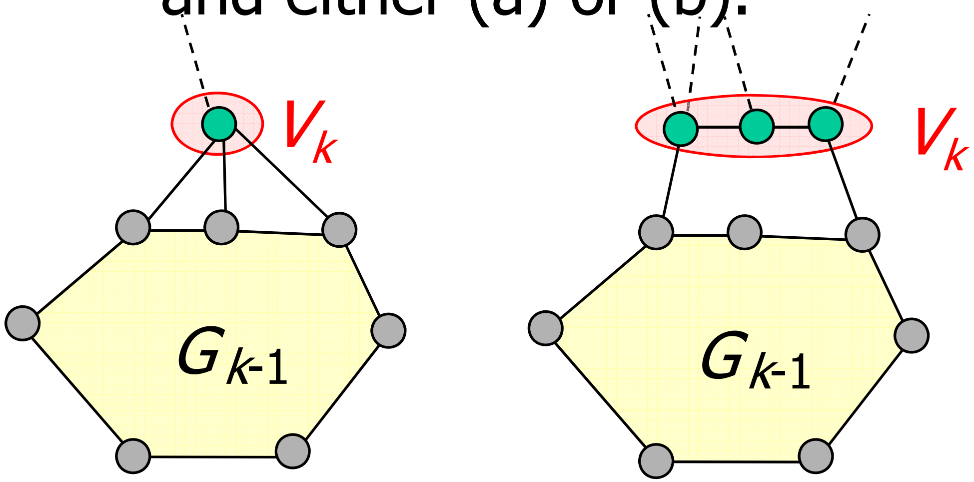
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$

⇒ $G$ has a canonical decomposition




(cd2) Each $G_k$ $(1 \le k \le h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \le k \le h-1)$ are outer vertices of $G_k$, and either (a) or (b).

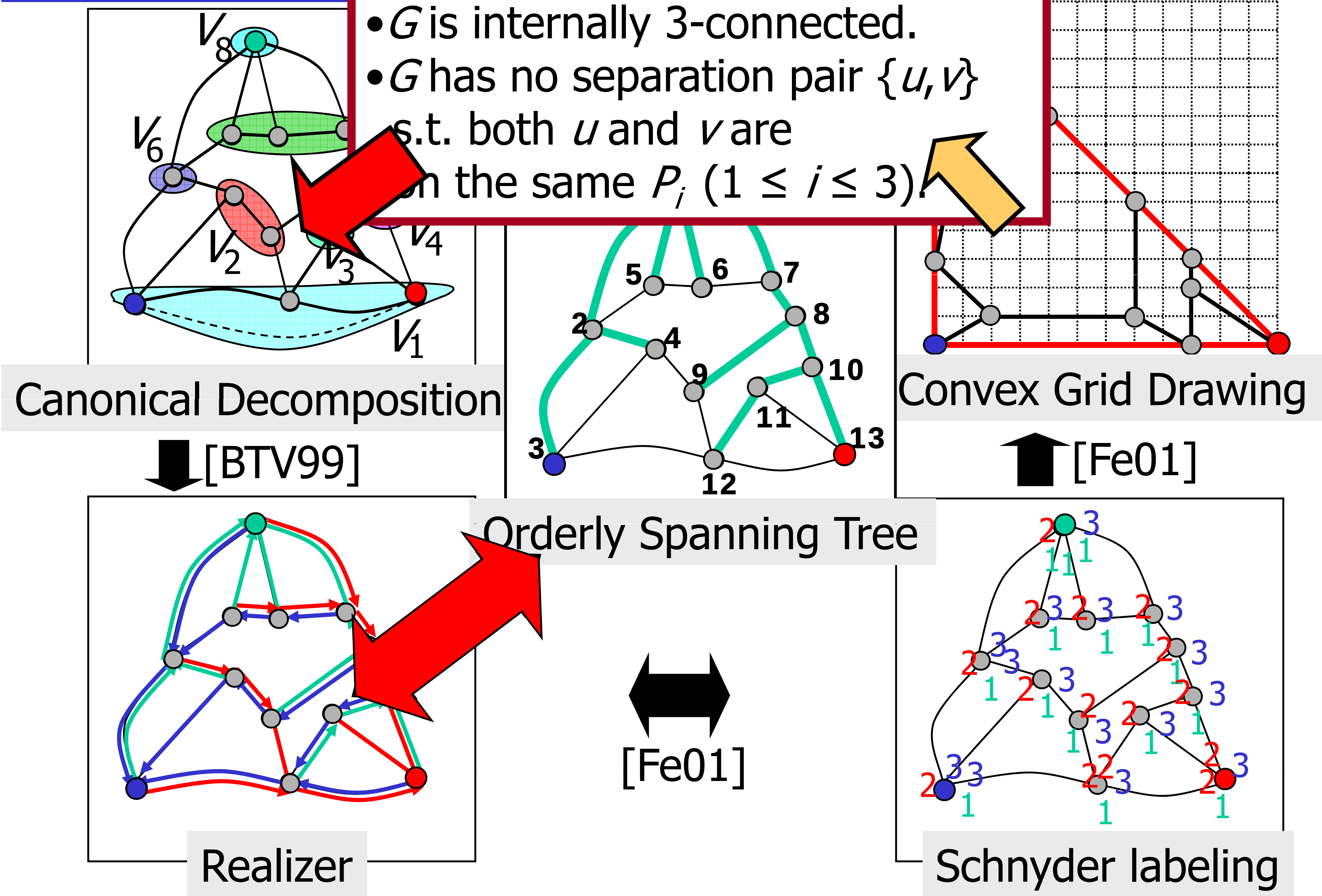



(a) (b)

# Our results

necessary and sufficient condition

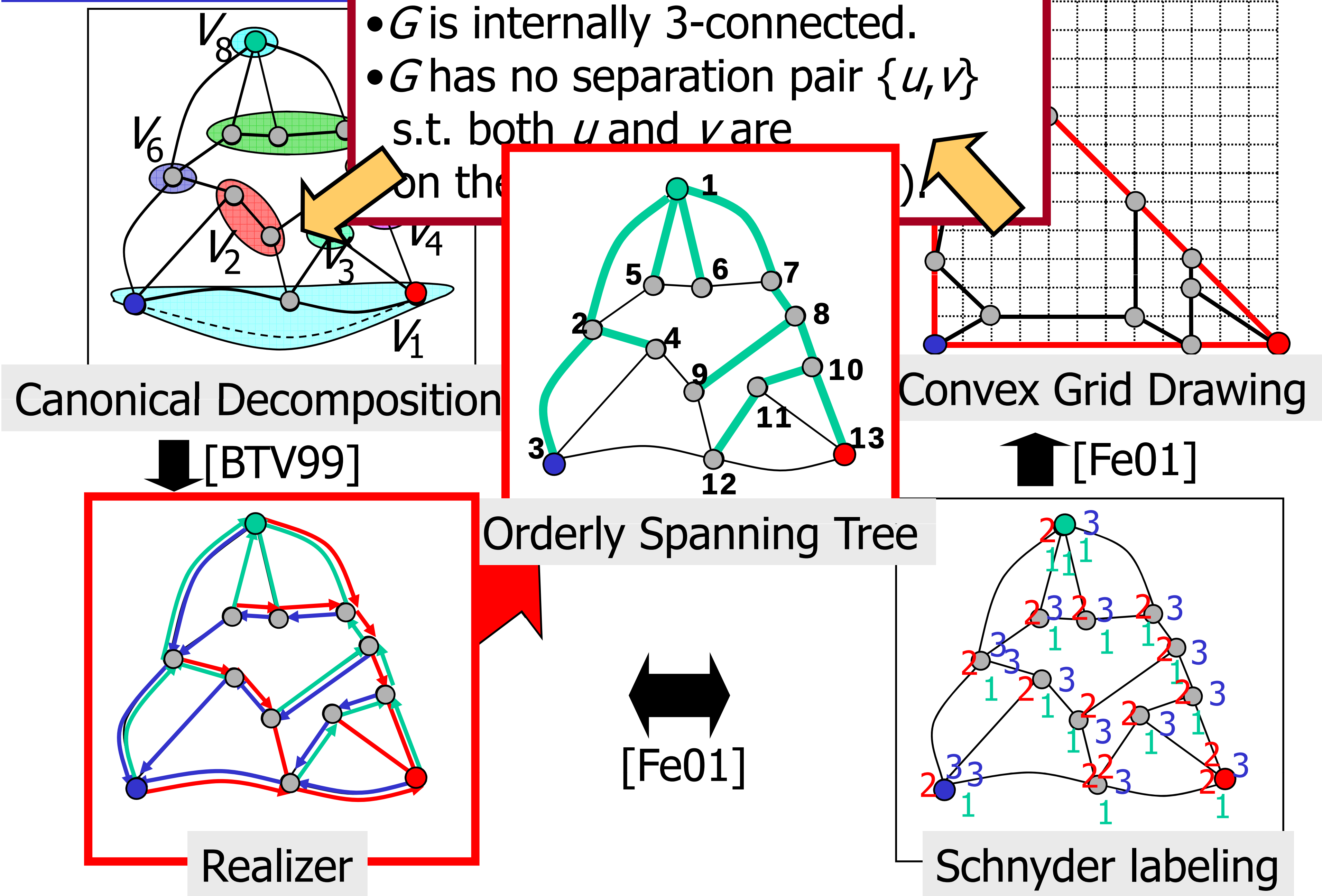
- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$.



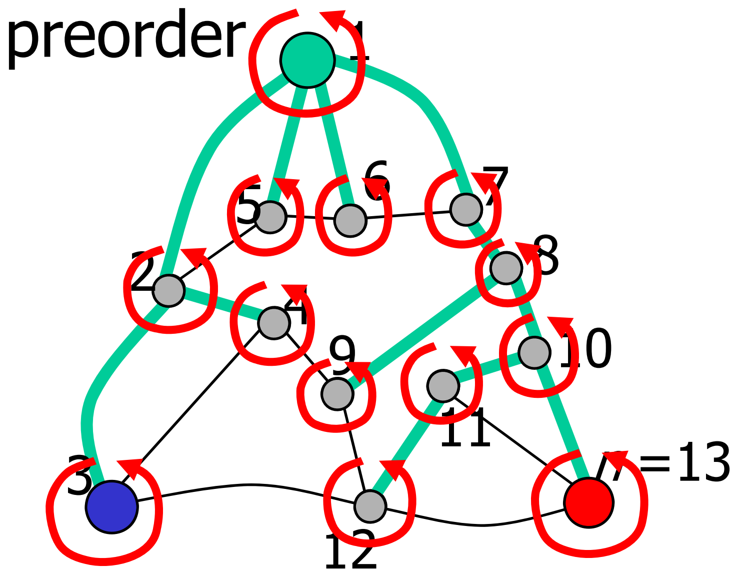
# Our results

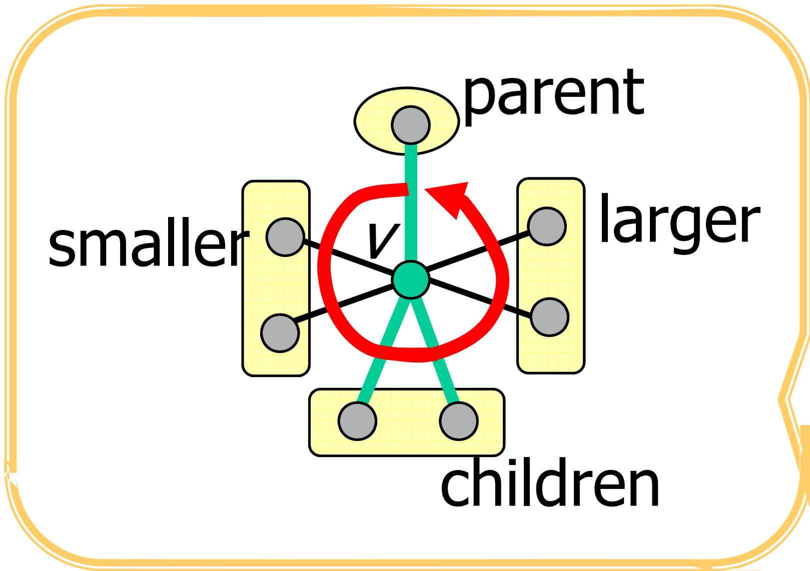
necessary and sufficient condition

- *G* is internally 3-connected.
- *G* has no separation pair $\{u, v\}$ s.t. both *u* and *v* are on the ... ).

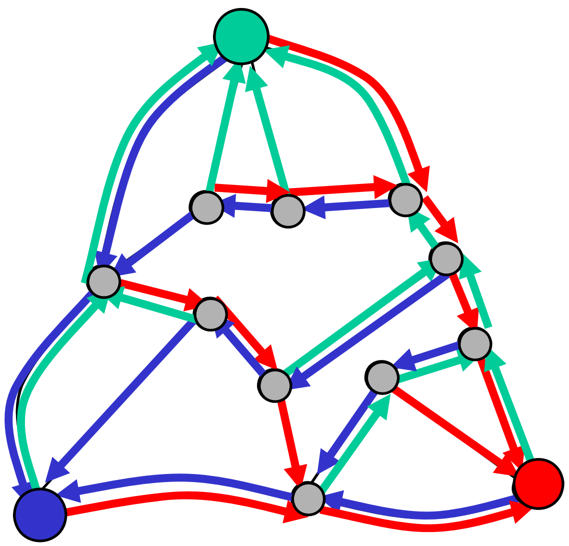


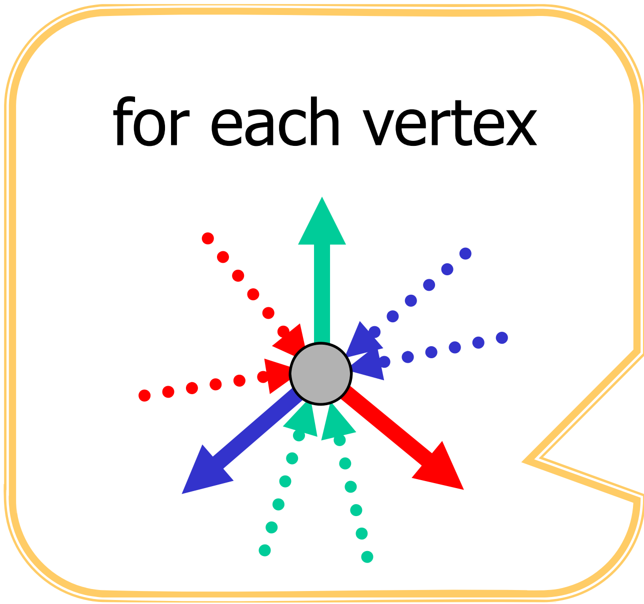
# Orderly Spanning Tree

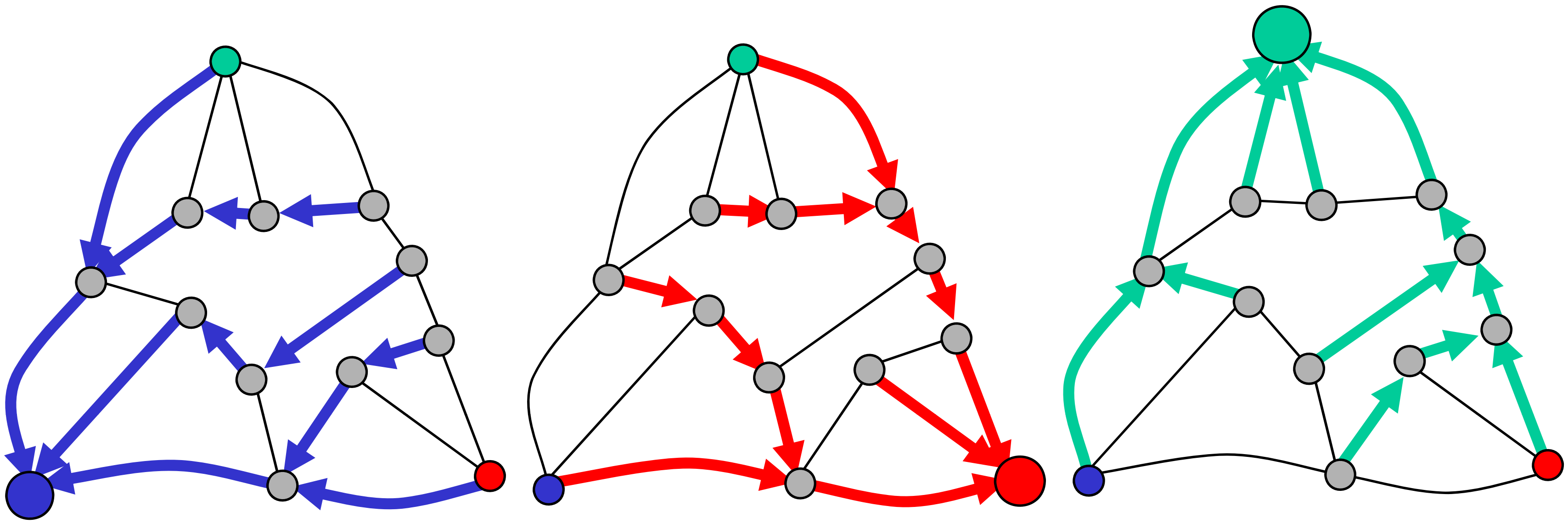


# Realizer



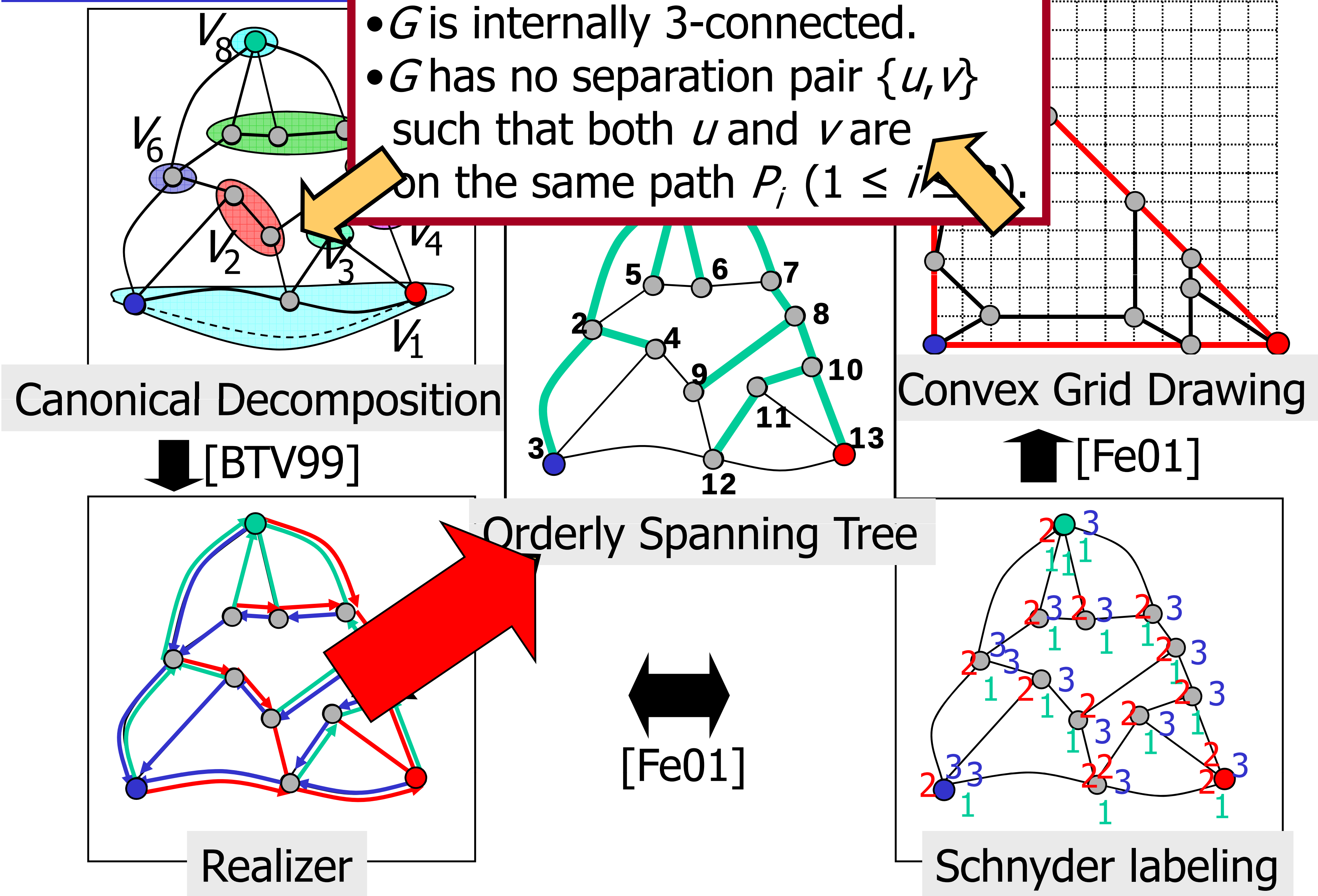


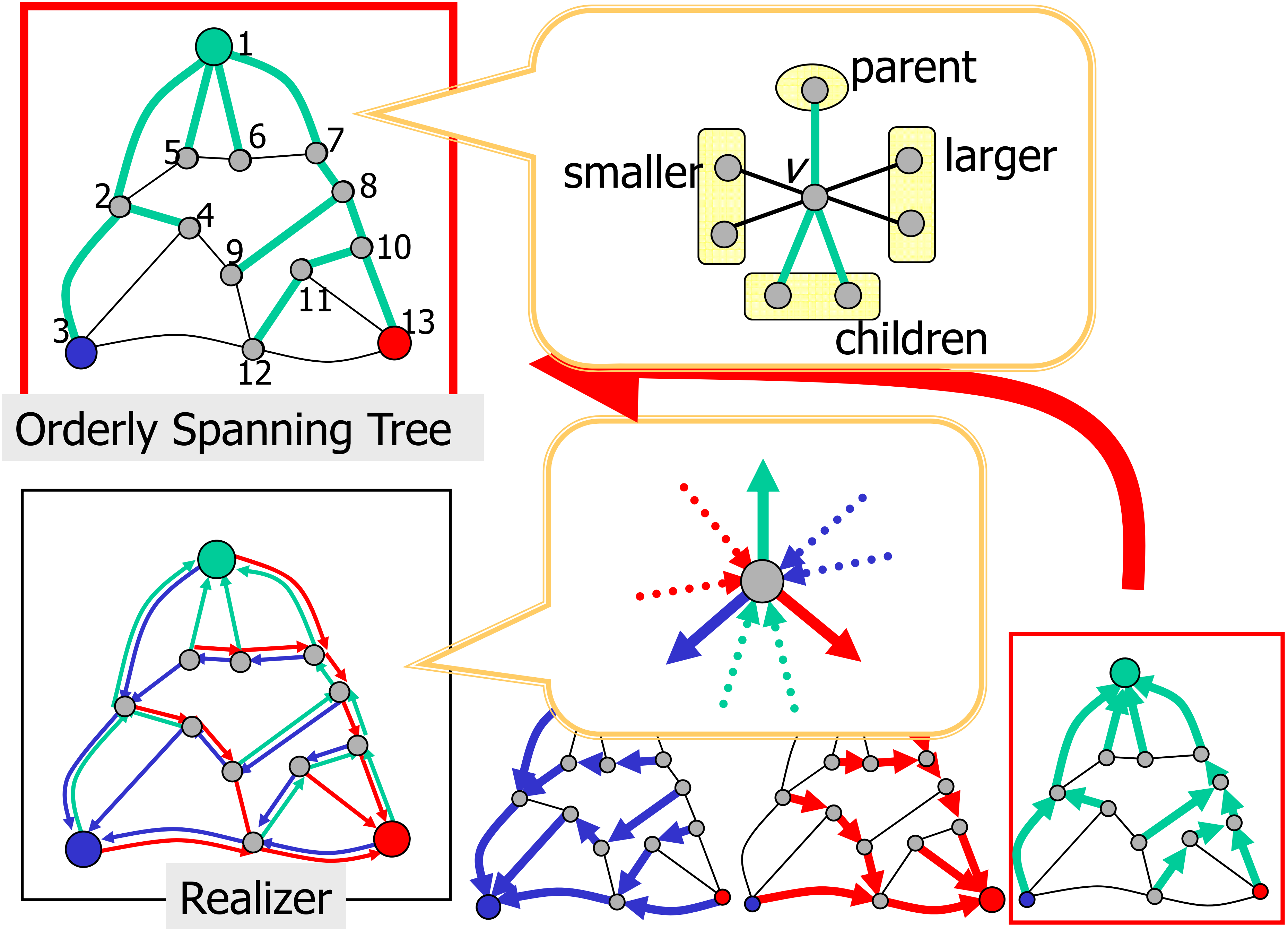



# Our results

necessary and sufficient condition

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u,v\}$ such that both $u$ and $v$ are on the same path $P_i$ ($1 \le i \le$ [illegible]).

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Orderly Spanning Tree
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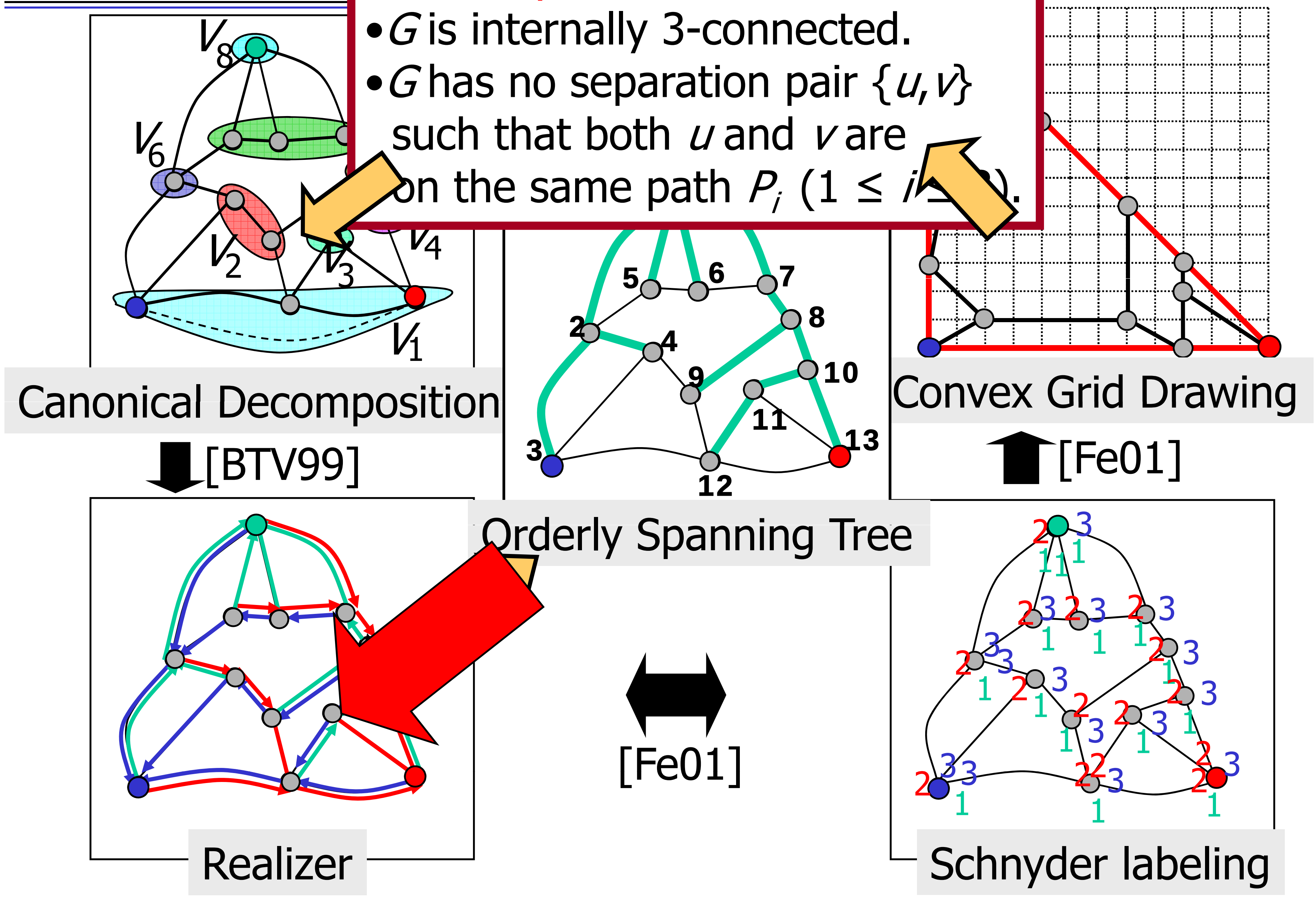
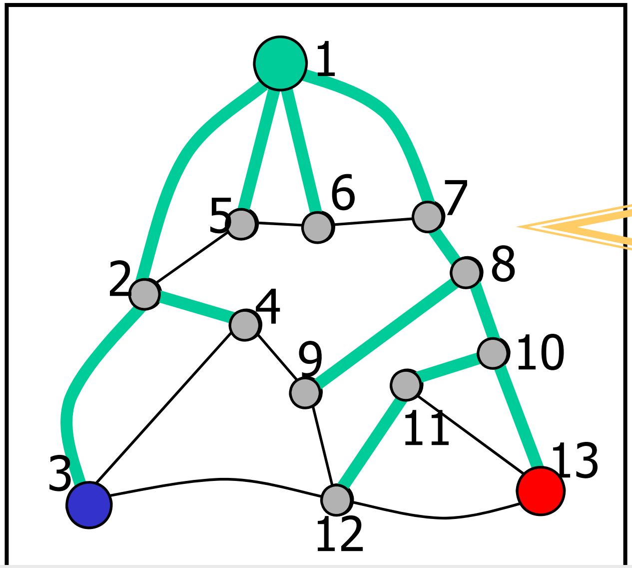
larger
children
Realizer

# Our results

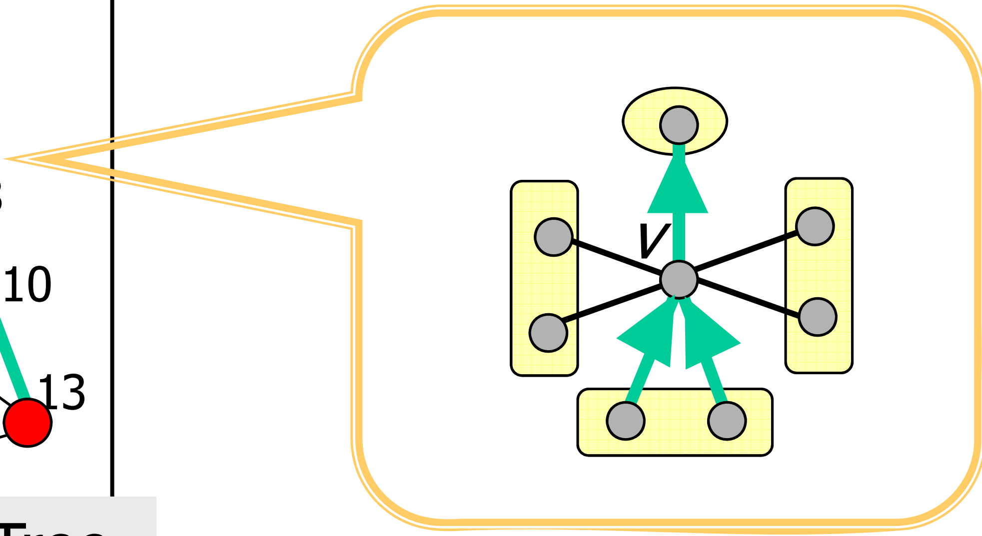
necessary and sufficient condition

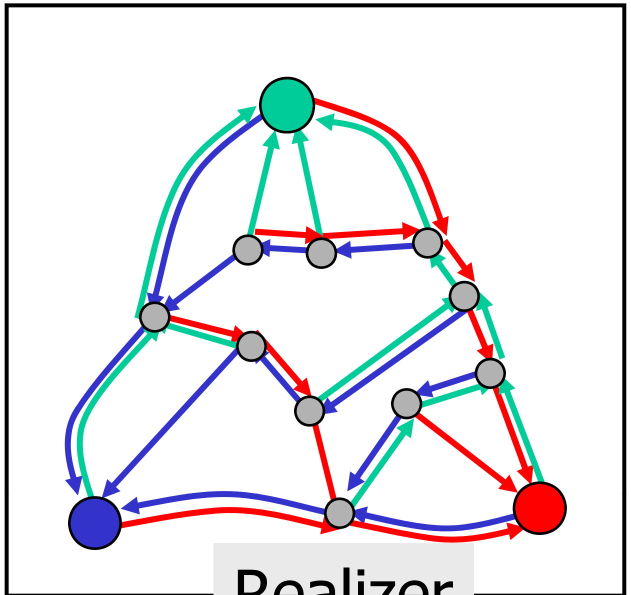
- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u,v\}$ such that both $u$ and $v$ are on the same path $P_i$ ($1 \le i \le$ 3).



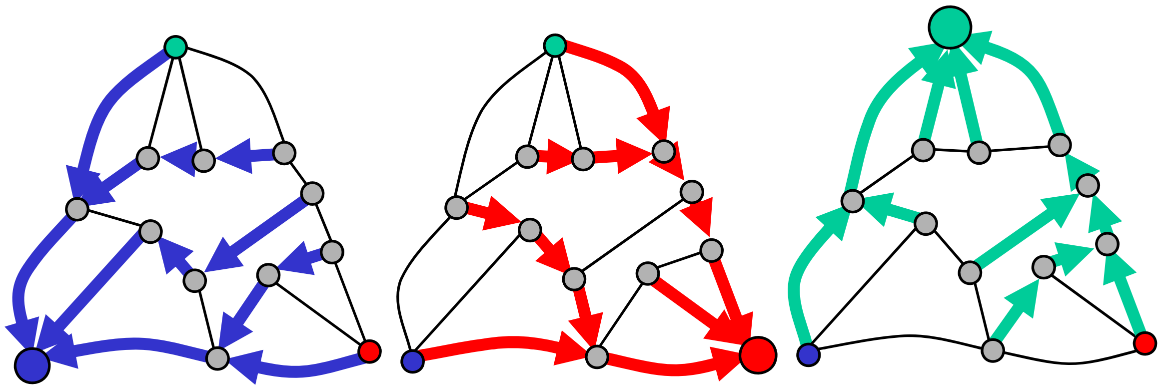


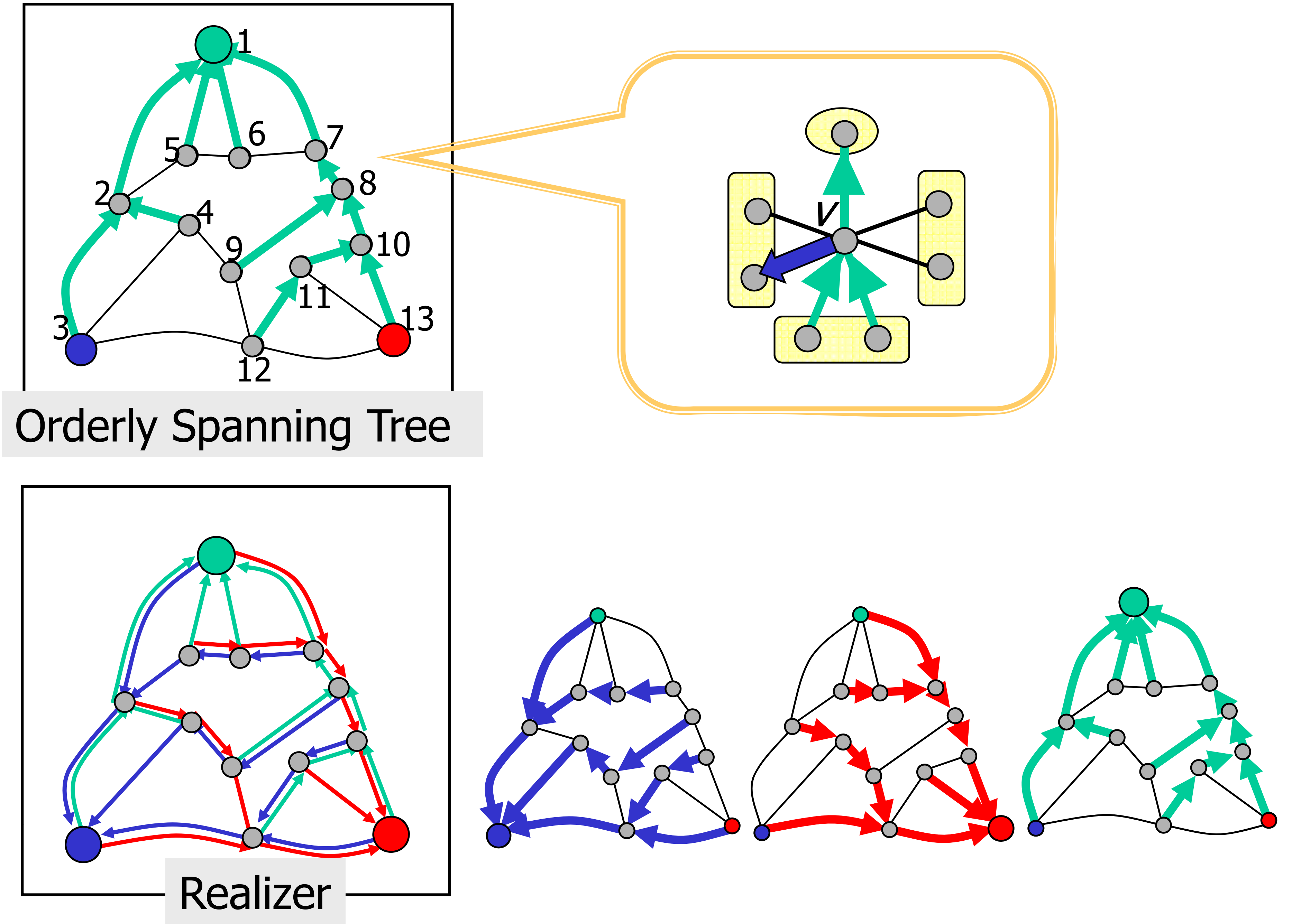

Orderly Spanning Tree



Realizer

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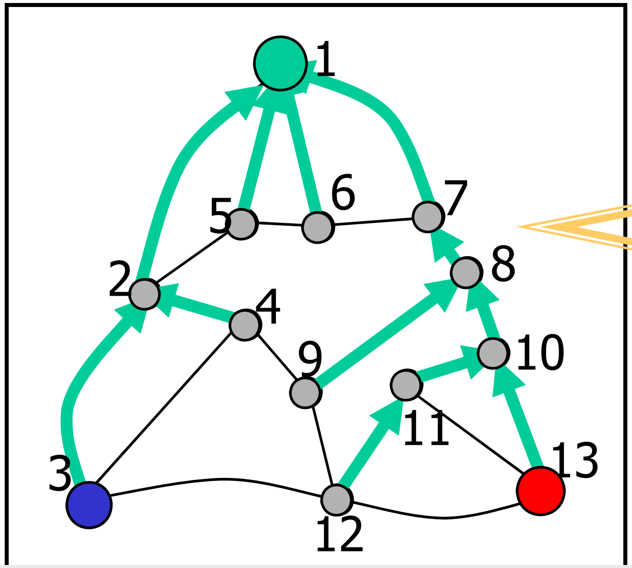
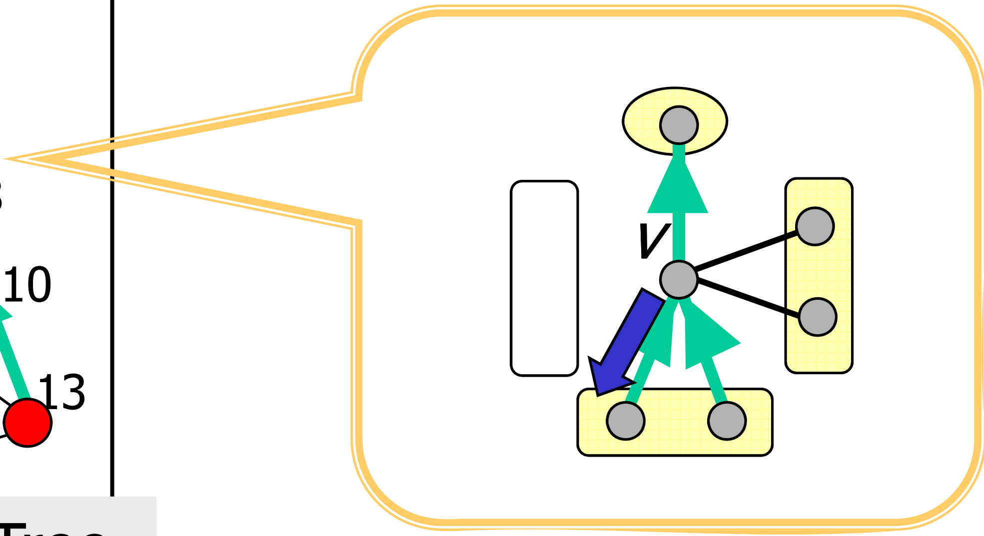
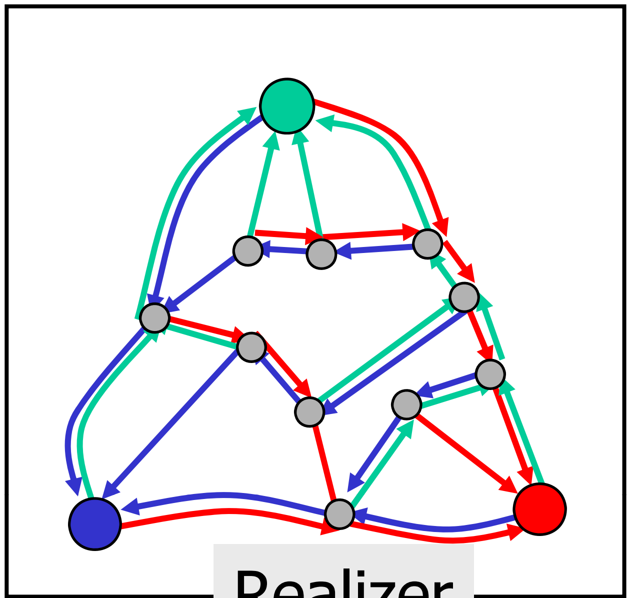
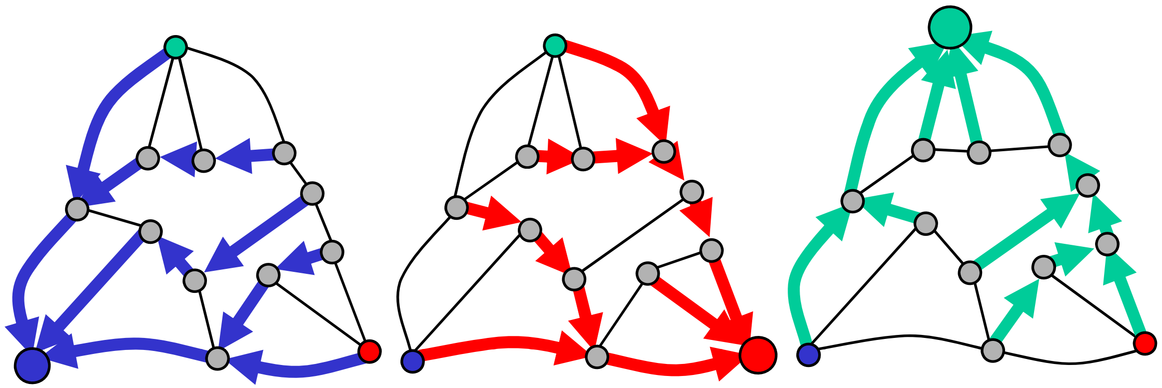


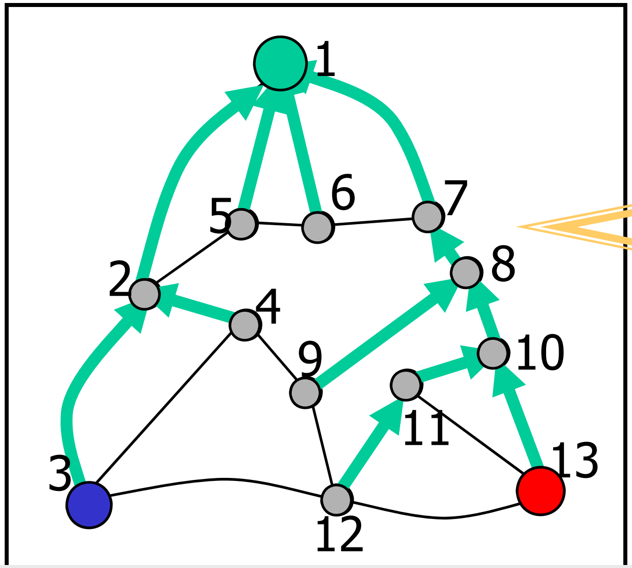

Orderly Spanning Tree



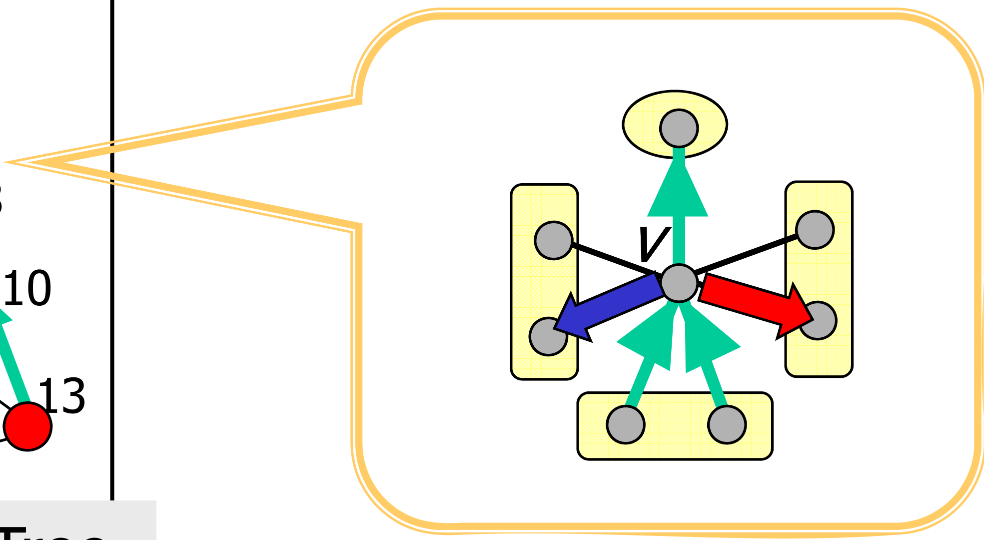



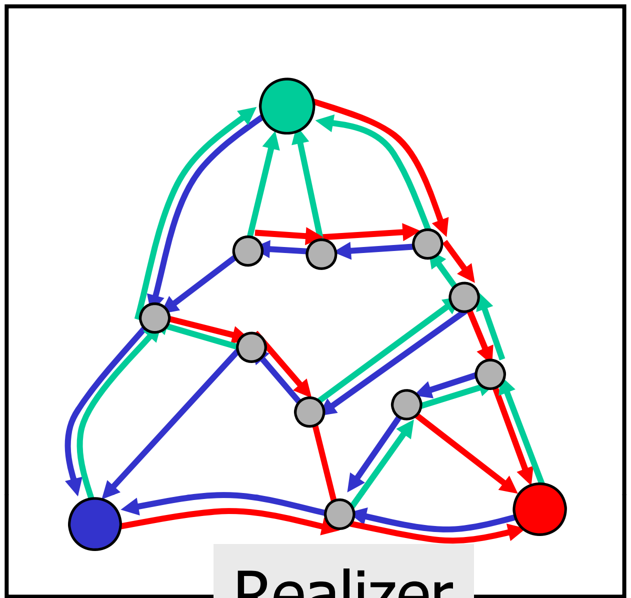
Realizer



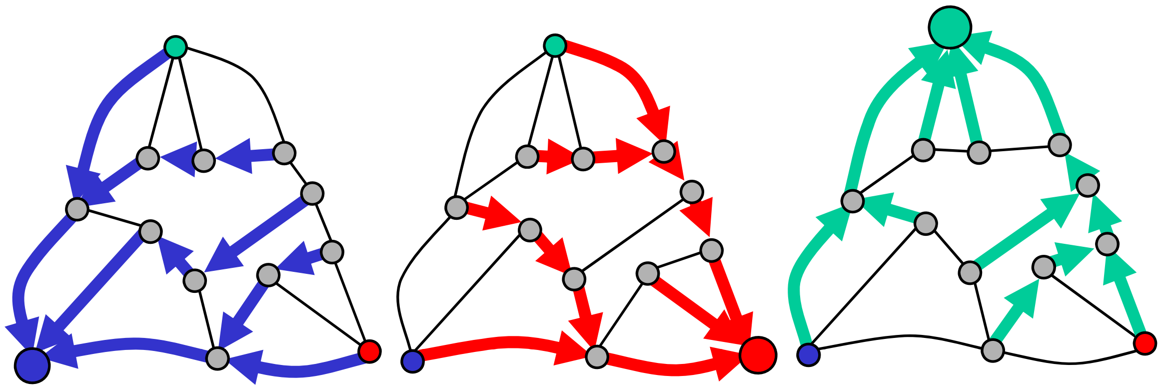


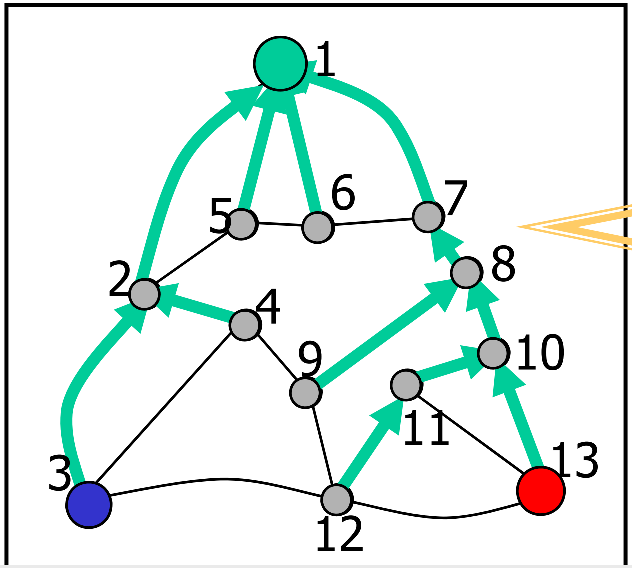

Orderly Spanning Tree



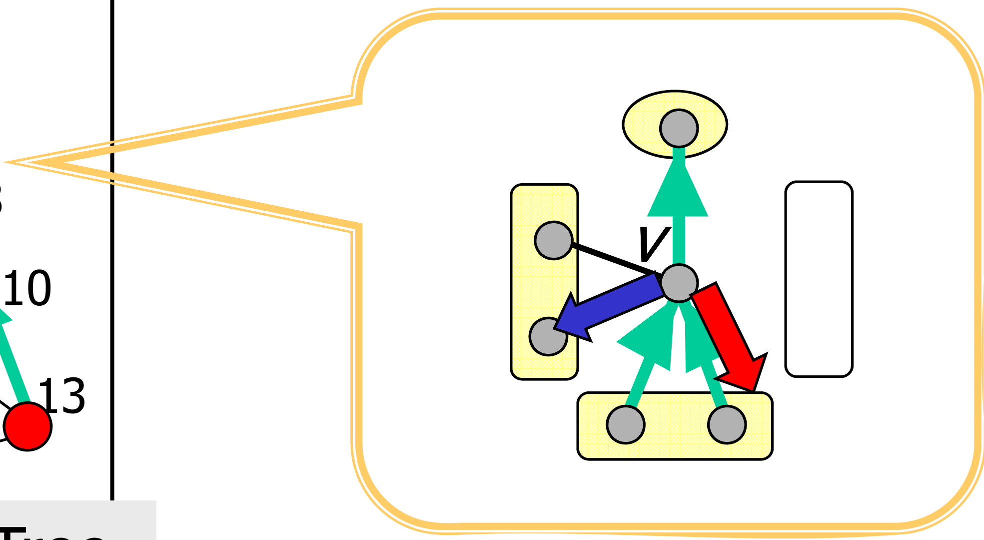



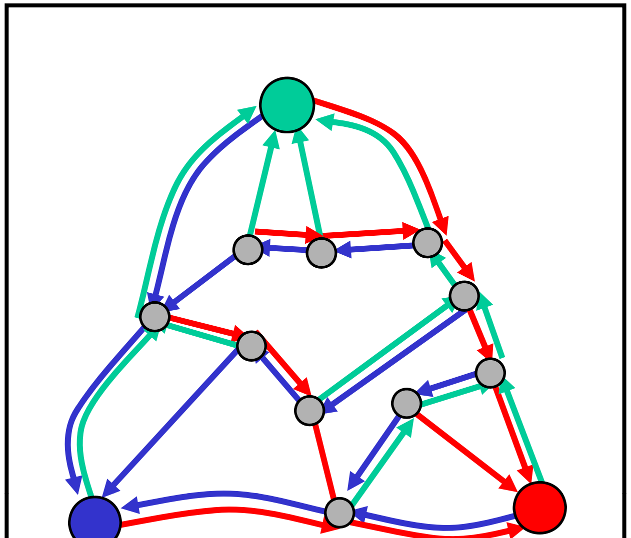
Realizer



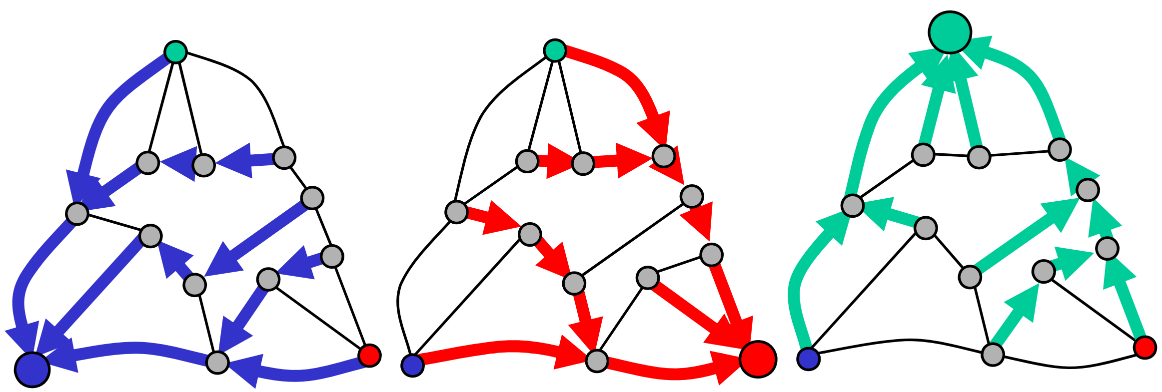


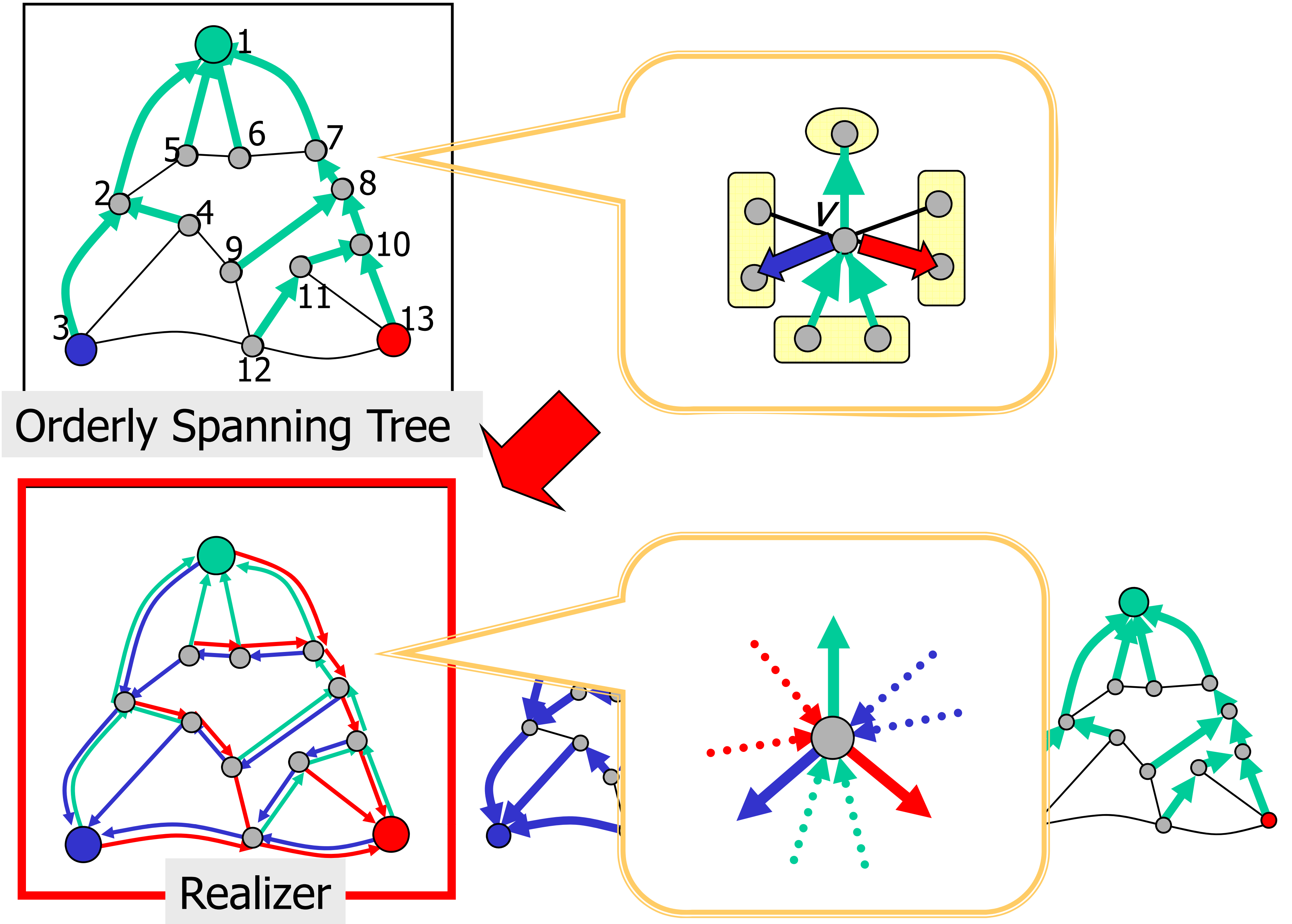

Orderly Spanning Tree



Realizer

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Orderly Spanning Tree
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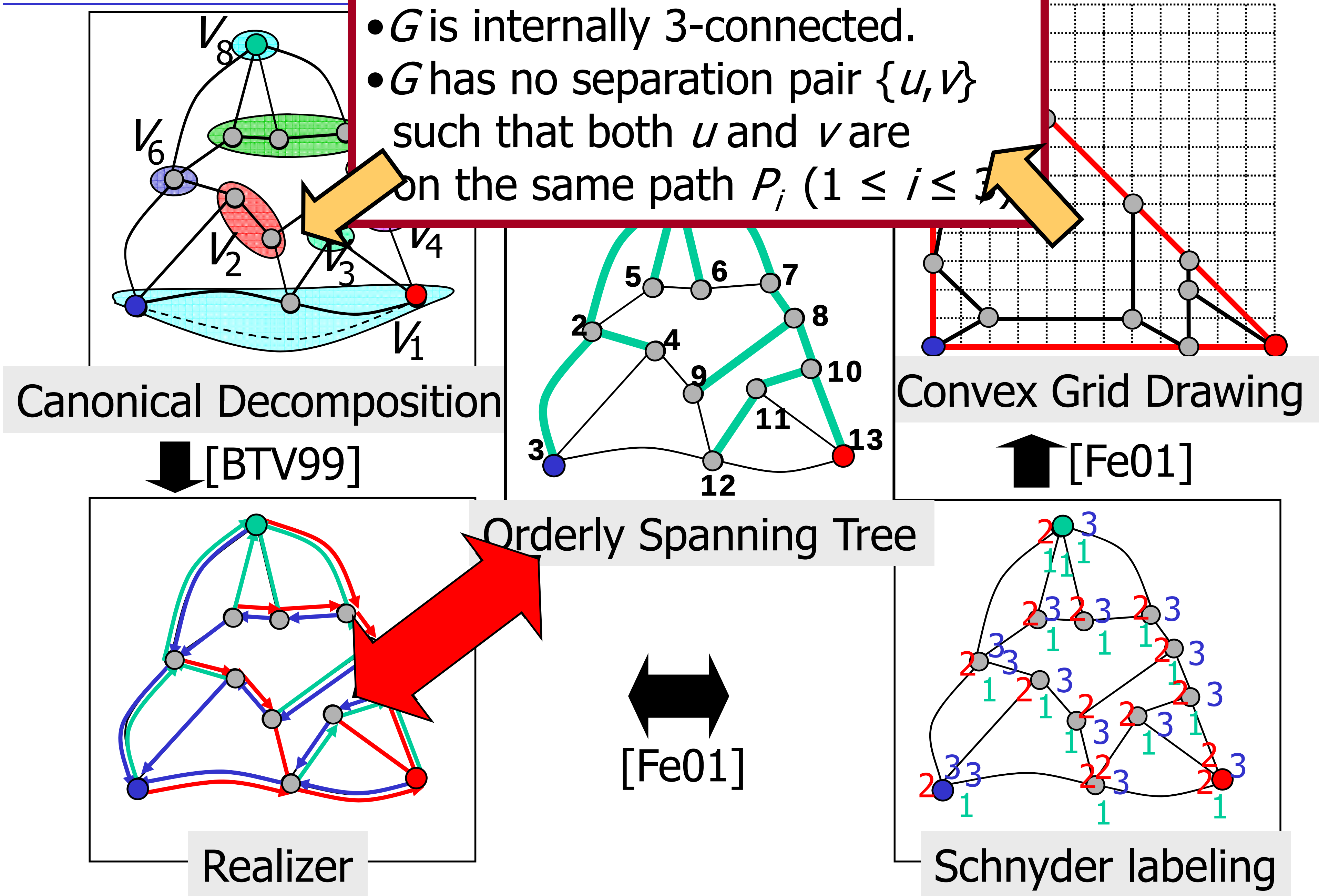
Realizer

# Conclusion

**necessary and sufficient condition**

- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same path $P_i$ $(1 \le i \le 3)$



# Theorem

$G$ : plane graph with each degree $\geq 3$

(a) - (f) are equivalent with each other.

(a) $G$ has a canonical decomposition.

(b) $G$ has a realizer.

(c) $G$ has a Schnyder labeling.

(d) $G$ has an outer triangular convex grid drawing.

(e) $G$ has an orderly spanning tree.

(f) necessary and sufficient condition

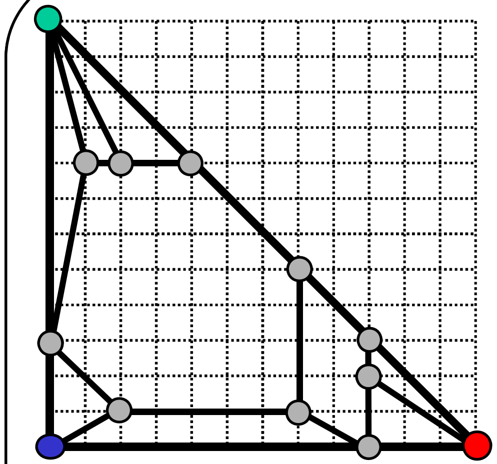
- $G$ is internally 3-connected.
- $G$ has no separation pair $\{u, v\}$ such that both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$.

# Corollary

*G* : plane graph with each degree ≥ 3

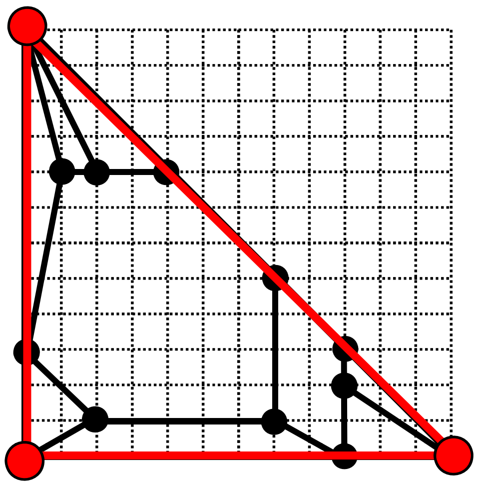
If a plane graph *G* satisfies
the necessary and sufficient condition,
then one can find the followings in linear time

(a) canonical decomposition,
(b) realizer,
(c) Schnyder labeling,
(d) orderly spanning tree, and
(e) outer triangular convex grid drawing
of *G* having the size $(n\text{-}1) \times (n\text{-}1)$.

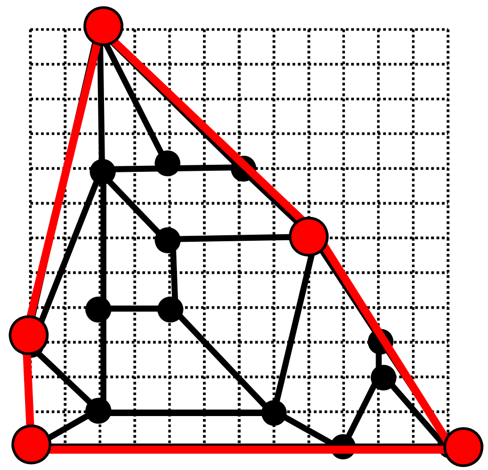


not 3-connected

**The remaining problem** is
to characterize the class of plane graphs
having convex grid drawings
such that the size is $(n-1) \times (n-1)$
and the outer face is not always a triangle.



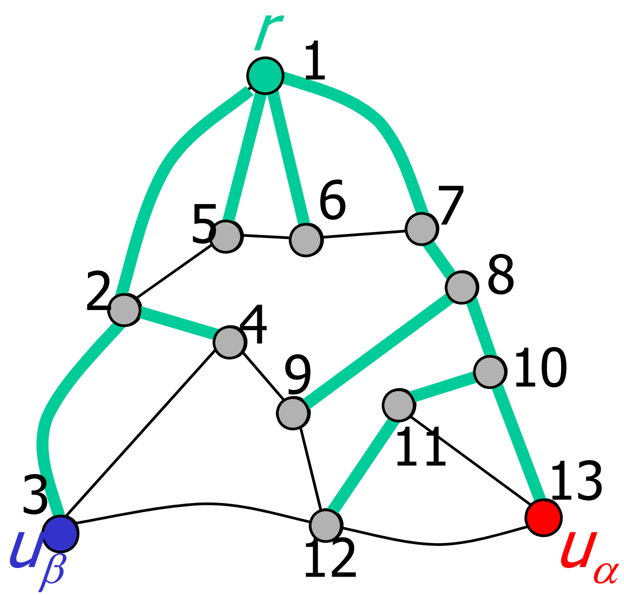
triangle

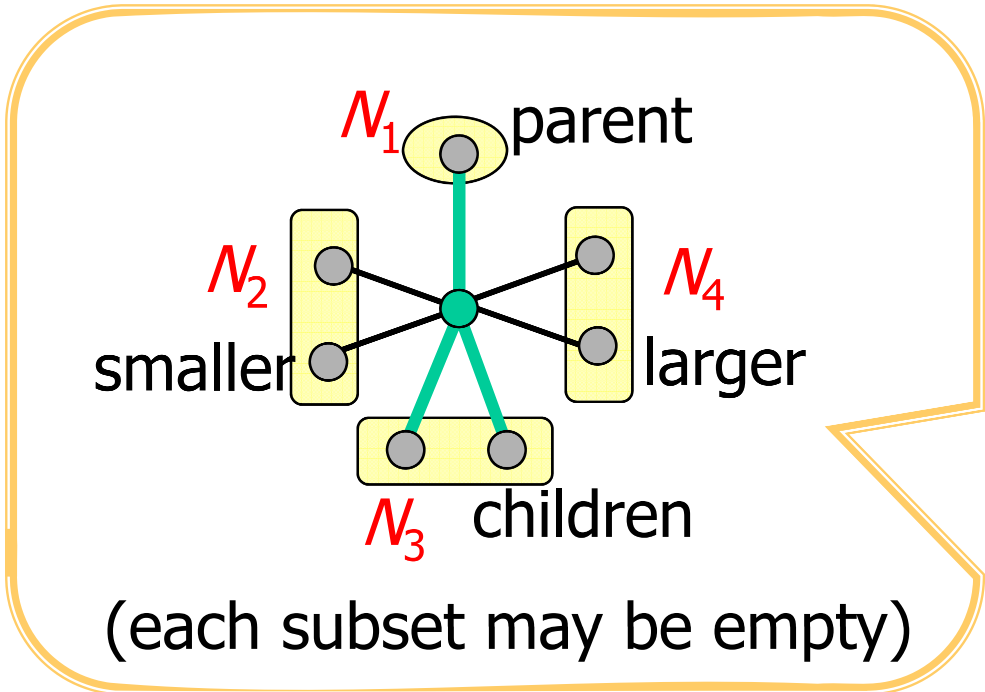


pentagon

END

# Orderly Spanning Tree




(ost1) For each edge not in the tree $T$,
none of the endpoints is an ancestor of the other in $T$.

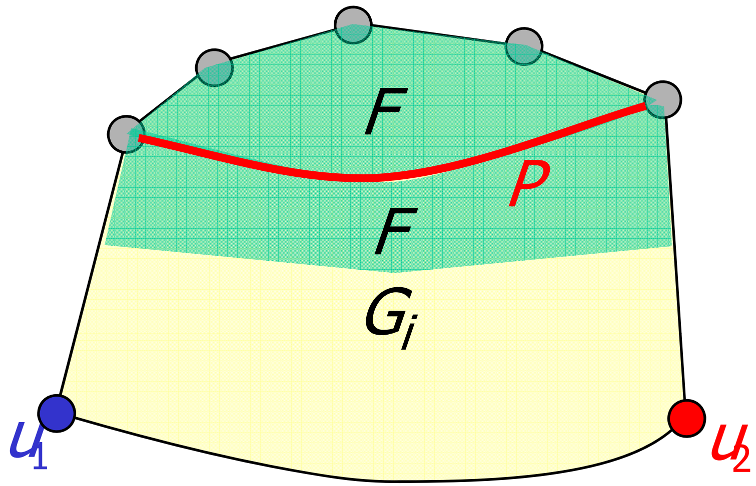
(ost2) For each leaf other than $u_\alpha$ and $u_\beta$,
neither $N_2$ nor $N_4$ is empty.

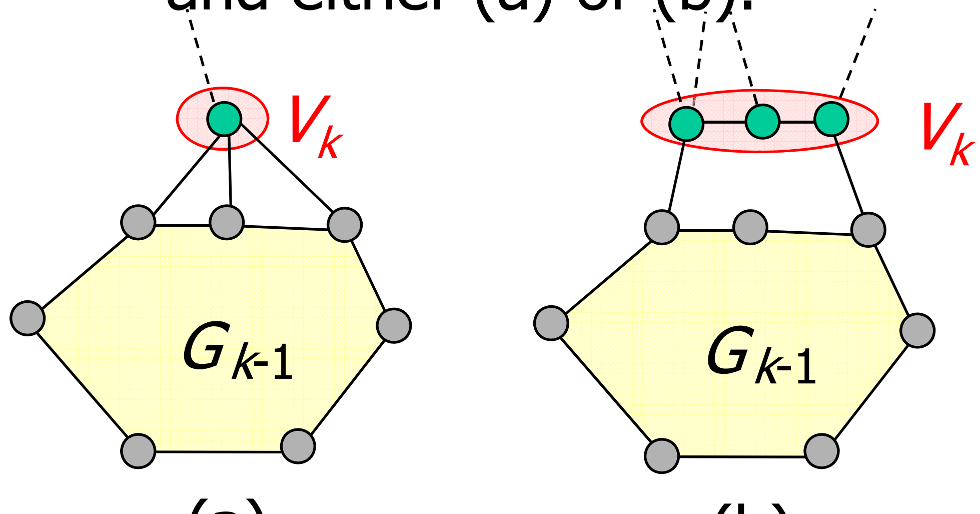
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

⇒ $G$ has a canonical decomposition

(cd2) Each $G_k$ $(1 \leq k \leq h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).




(a) (b)

- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$

⇒

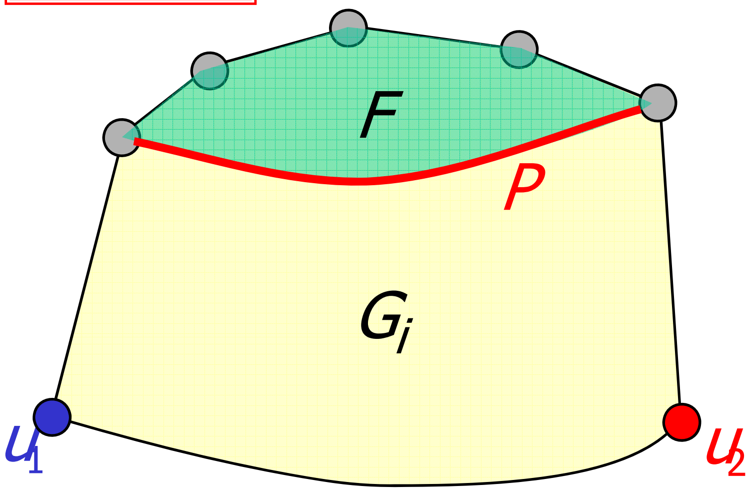
$G$ has a canonical decomposition

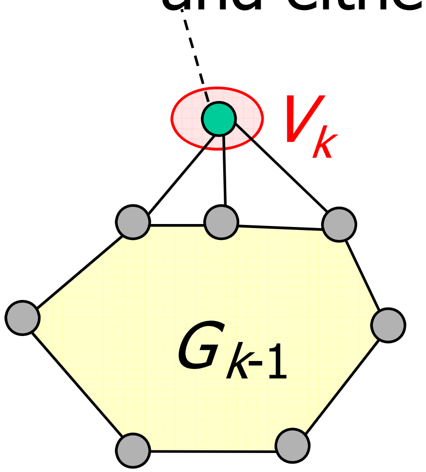
---

(cd2) Each $G_k$ $(1 \le k \le h)$ is internally 3-connected.

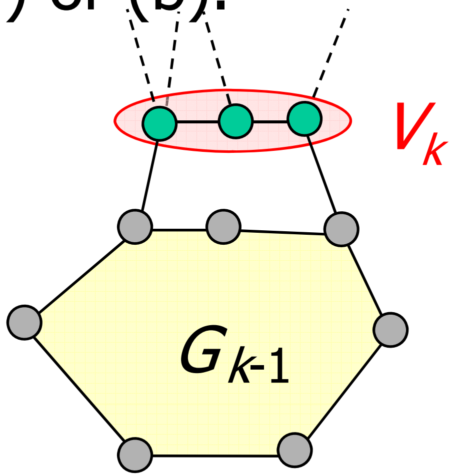
(cd3) All the vertices in each $V_k$ $(2 \le k \le h-1)$ are outer vertices of $G_k$, and either (a) or (b).

Case 1




(a)




(b)

- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

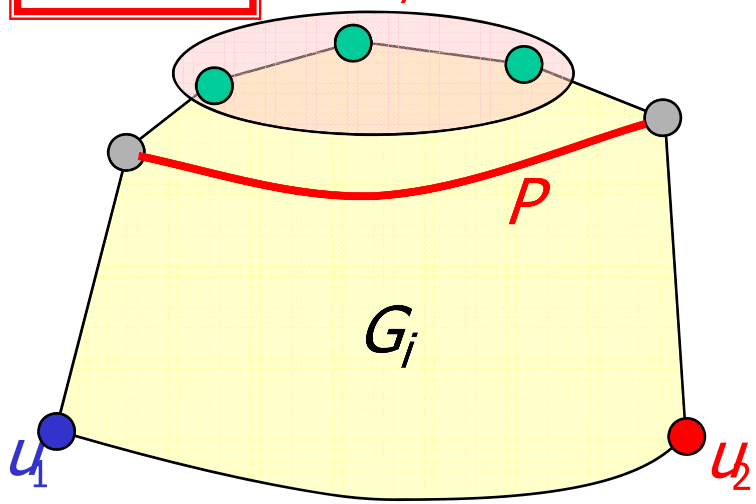
➡ $G$ has a canonical decomposition

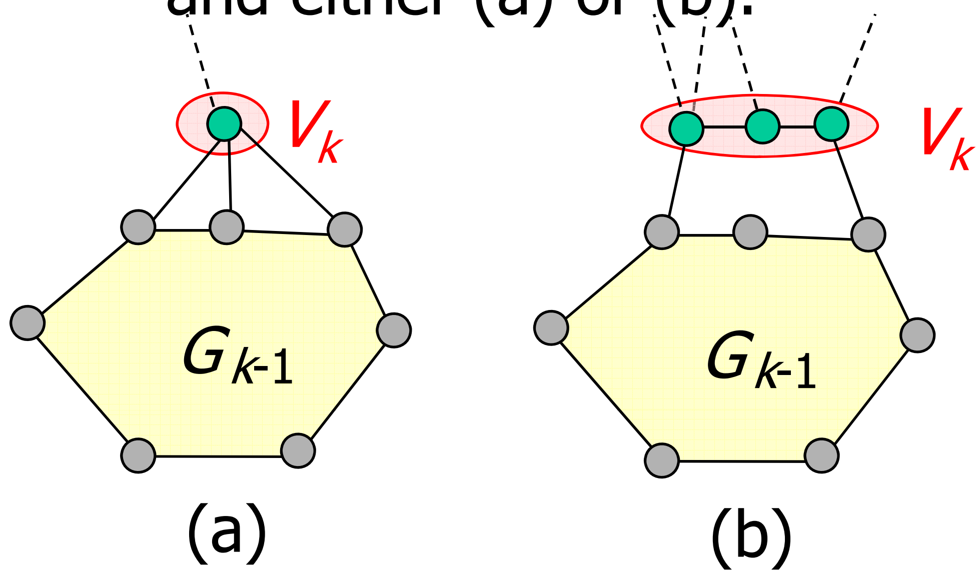
---

(cd2) Each $G_k$ $(1 \leq k \leq h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).

Case 1



- $G$ is internally 3-connected
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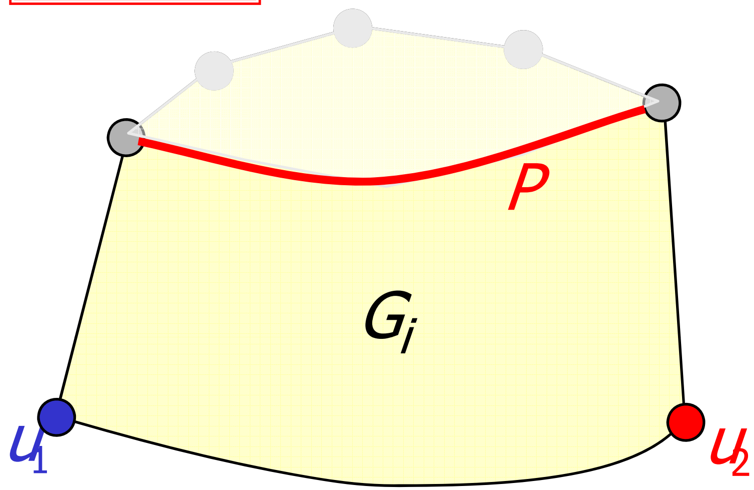
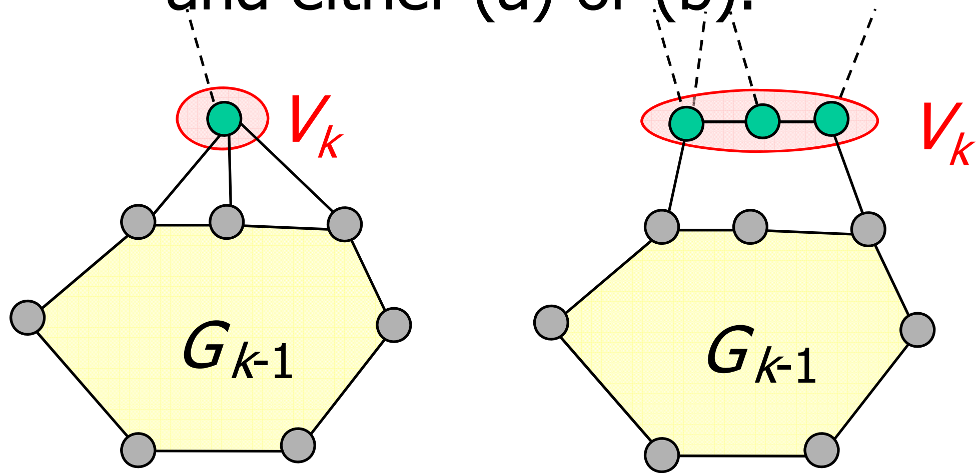
⇒ $G$ has a canonical decomposition

(cd2) Each $G_k$ $(1 \leq k \leq h)$ is ○ internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).

Case 1




(a) (b)

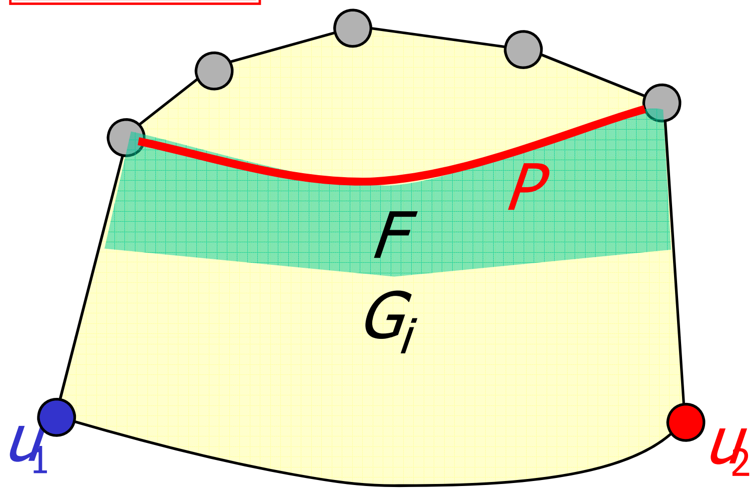
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$

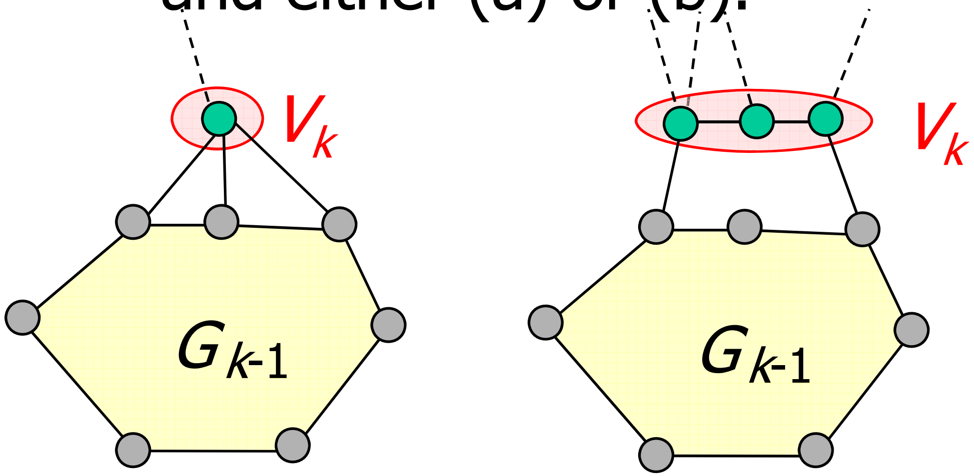
➡ $G$ has a canonical decomposition

(cd2) Each $G_k$ $(1 \le k \le h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \le k \le h-1)$ are outer vertices of $G_k$, and either (a) or (b).

Case 2



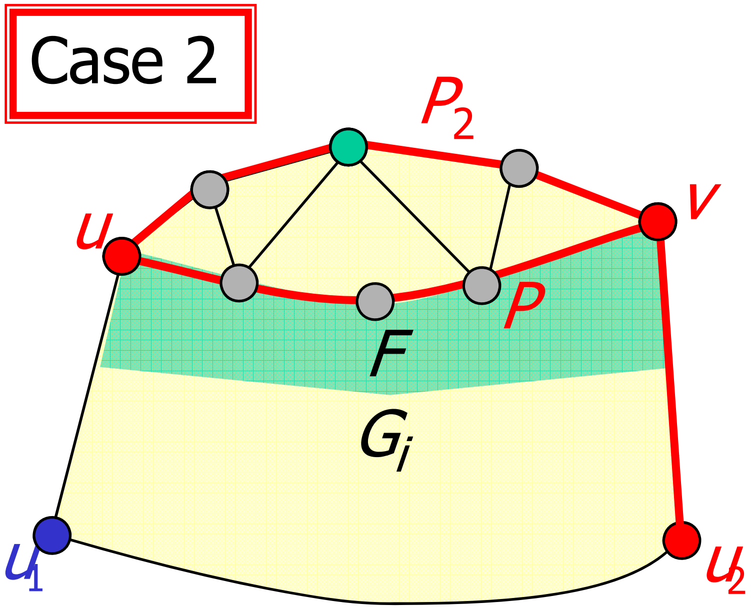

(a) (b)

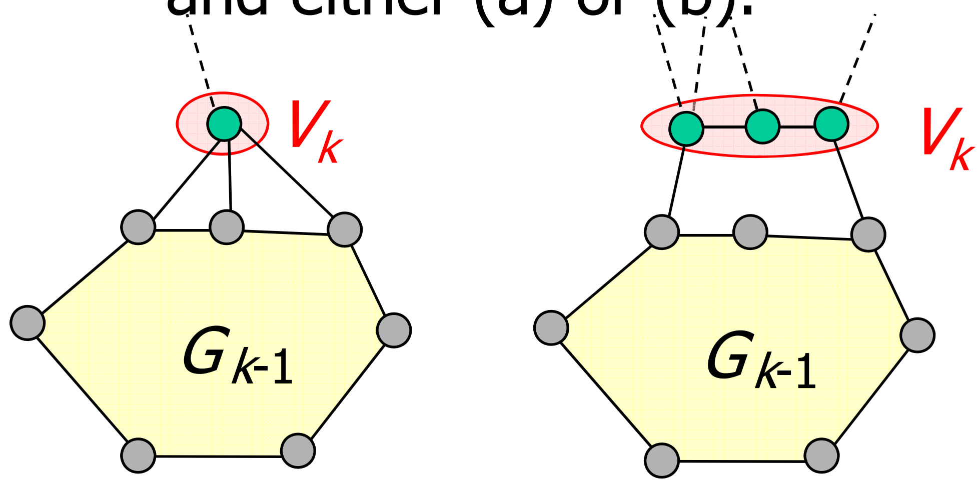
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

⇒ $G$ has a canonical decomposition

(cd2) Each $G_k$ $(1 \leq k \leq h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).




(a) (b)

- $G$ is internally 3-connected
- ~~$G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \le i \le 3$)~~

➡

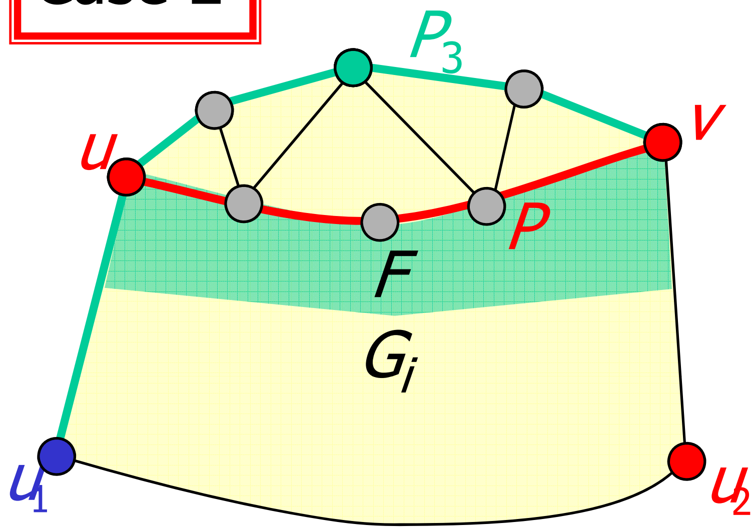
$G$ has a canonical decomposition

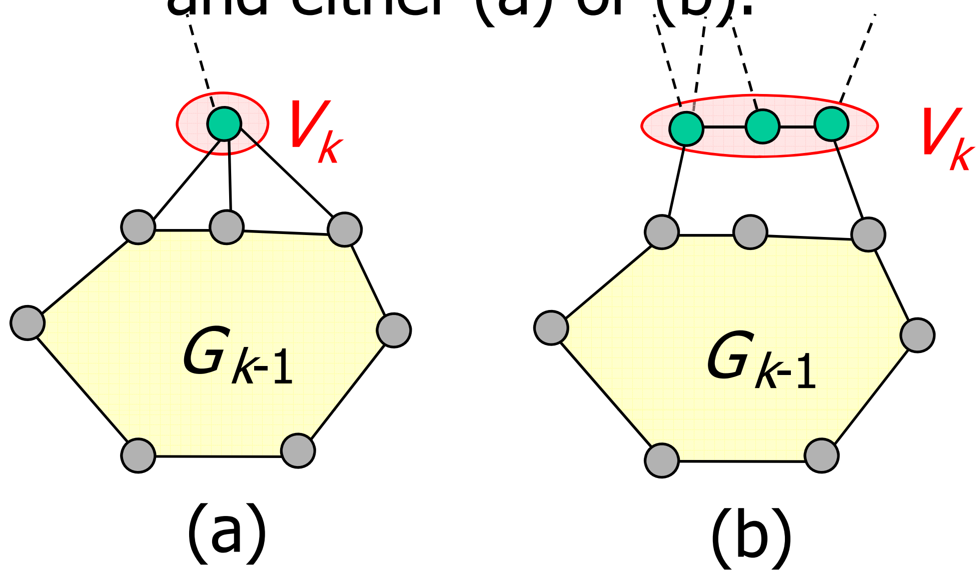
---

(cd2) Each $G_k$ ($1 \le k \le h$) is internally 3-connected.

(cd3) All the vertices in each $V_k$ ($2 \le k \le h-1$) are outer vertices of $G_k$, and either (a) or (b).

Case 2



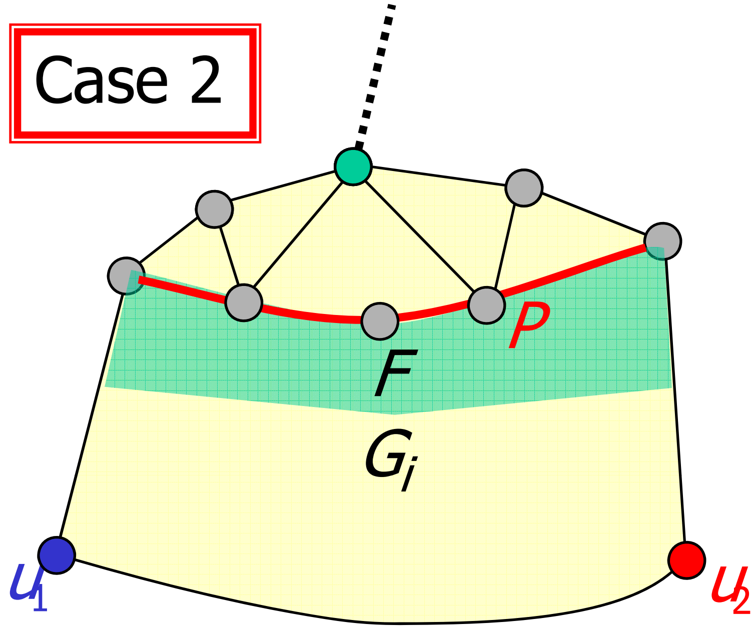
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$

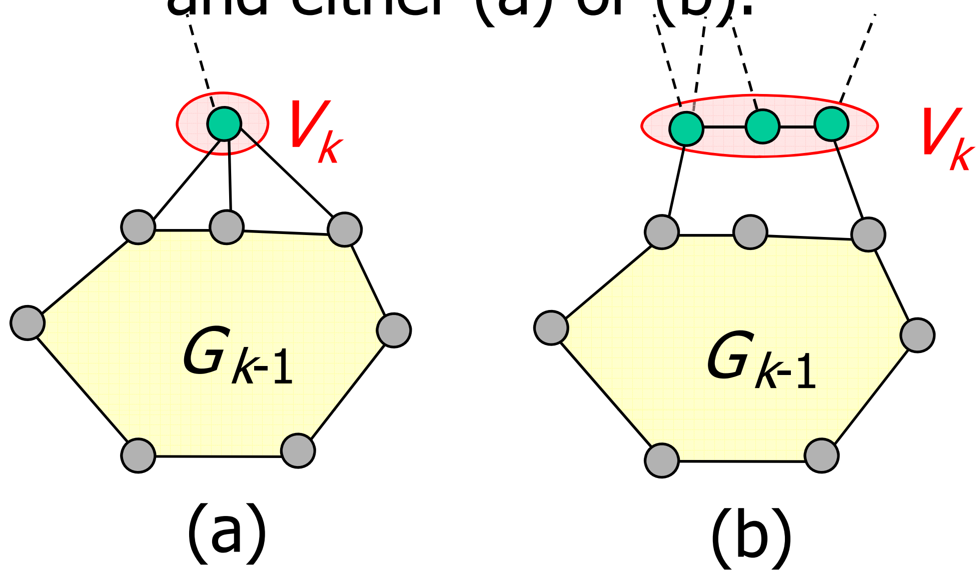
⇒ $G$ has a canonical decomposition

(cd2) Each $G_k$ $(1 \le k \le h)$ is internally 3-connected.

(cd3) All the vertices in each $V_k$ $(2 \le k \le h-1)$ are outer vertices of $G_k$, and either (a) or (b).

Case 2



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- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \le i \le 3)$

⇒ $G$ has a canonical decomposition

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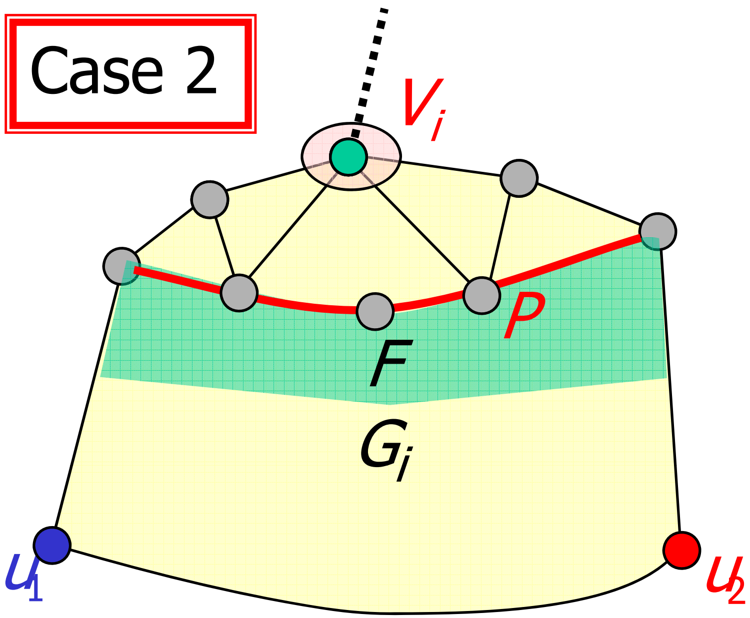
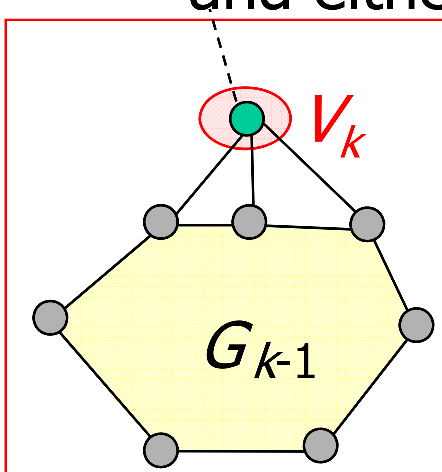
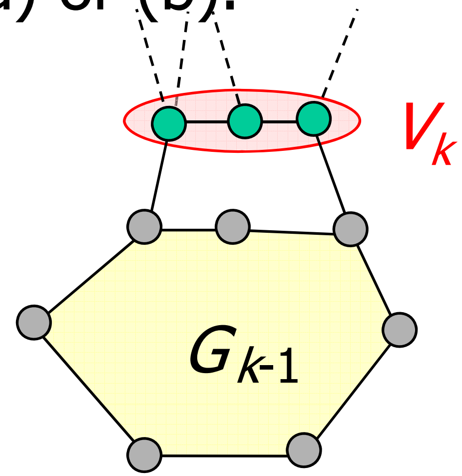
(cd3) All the vertices in each $V_k$ $(2 \le k \le h-1)$ are outer vertices of $G_k$, and either (a) or (b).

Case 2




(a)




(b)

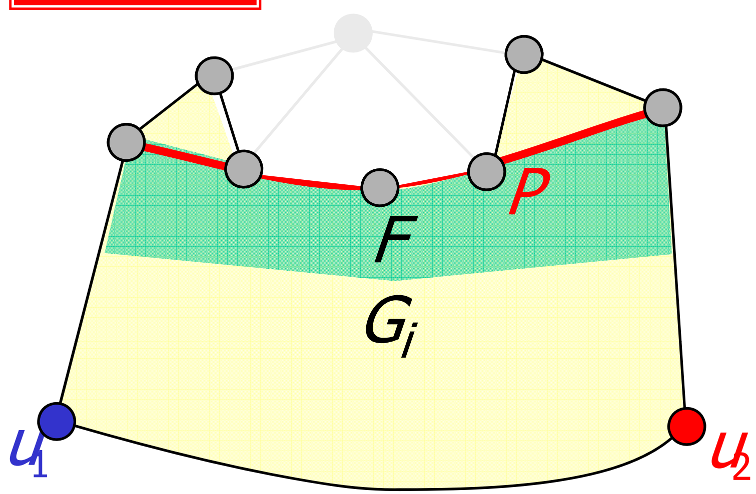
- $G$ is internally 3-connected
- $G$ has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

⇒

$G$ has a canonical decomposition

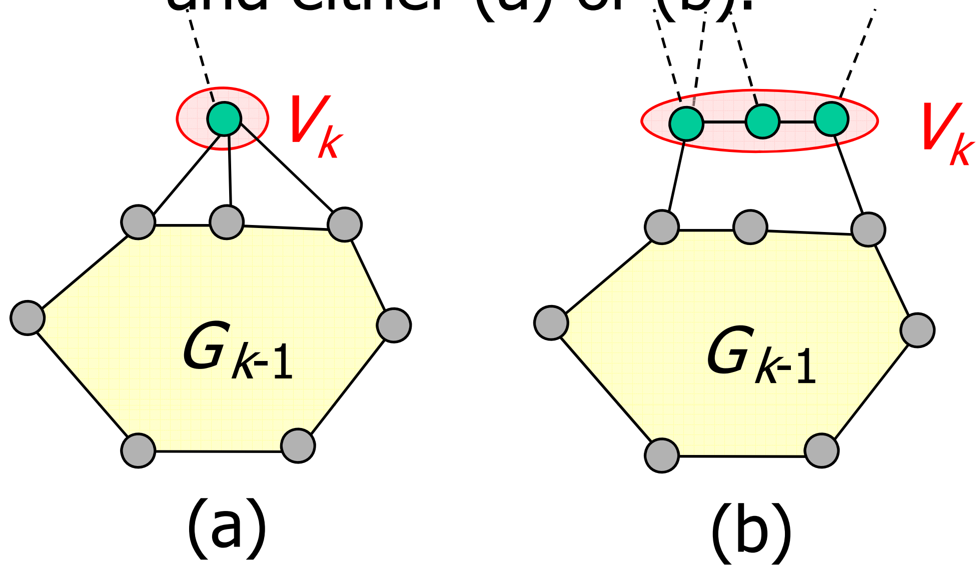
---

Case 2




(cd2) Each $G_k$ $(1 \leq k \leq h)$ is ○ internally 3-connected.

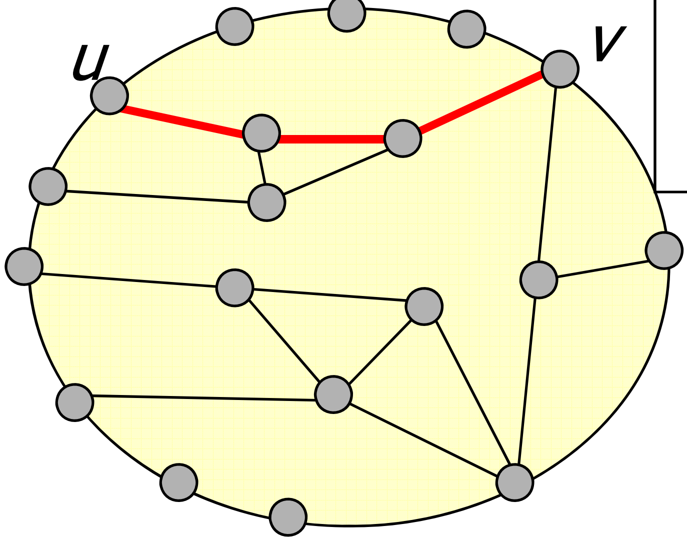
(cd3) All the vertices in each $V_k$ $(2 \leq k \leq h-1)$ are outer vertices of $G_k$, and either (a) or (b).



- $G$ is internally 3-connected
- G has no separation pair $\{u,v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \le i \le 3$)

⇒ $G$ has a canonical decomposition

chord-path $P$




1. $P$ connects two outer vertices $u$ and $v$.
2. $u, v$ is a separation pair of $G$.
3. $P$ lies on an inner face.
4. $P$ does not pass through any outer edge and any outer vertex other than $u$ and $v$.

- $G$ is internally 3-connected
- G has no separation pair $\{u, v\}$ s.t. both $u$ and $v$ are on the same $P_i$ ($1 \leq i \leq 3$)

⇒ $G$ has a canonical decomposition

1. $P$ connects two outer vertices $u$ and $v$.
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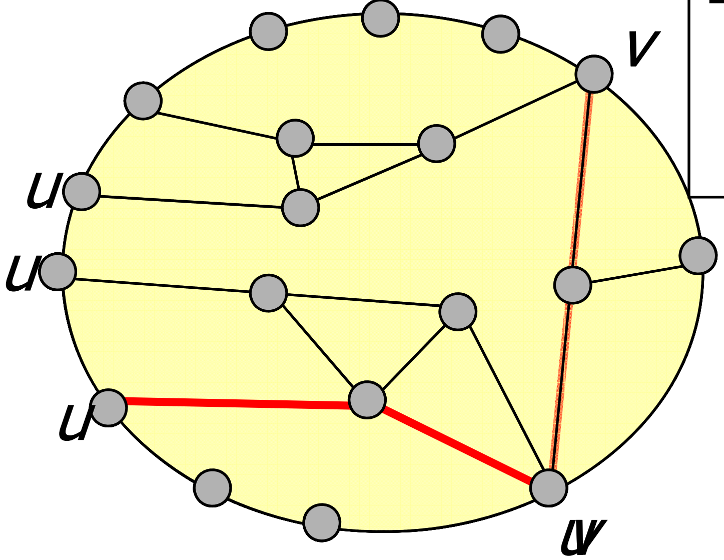
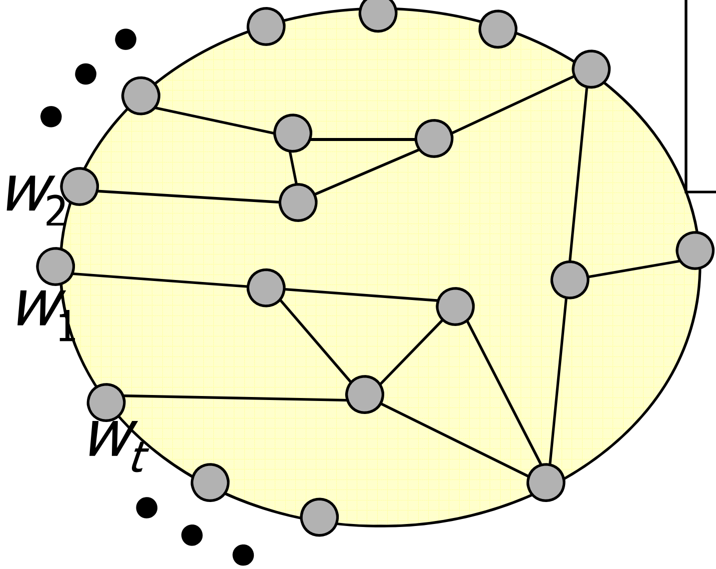
chord-path $P$



- $G$ is internally 3-connected
- G has no separation pair $\{u,v\}$ s.t. both $u$ and $v$ are on the same $P_i$ $(1 \leq i \leq 3)$

⇒ $G$ has a canonical decomposition

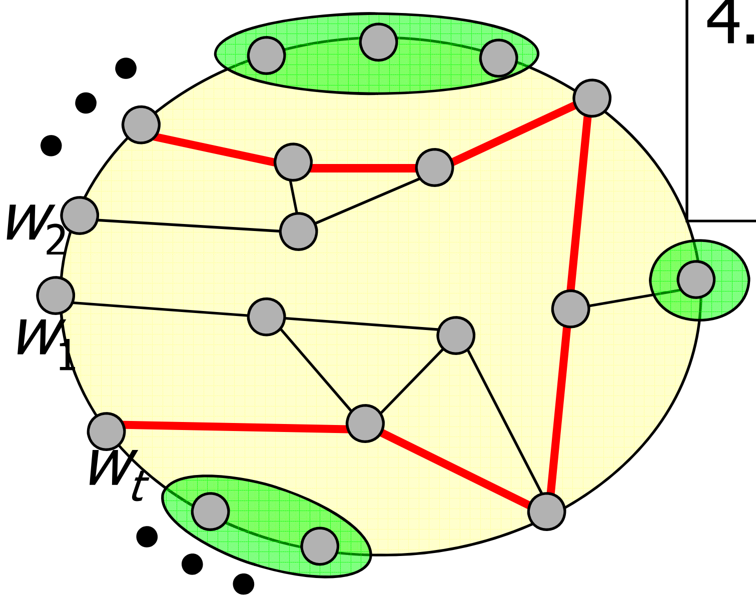
minimal chord-path $P$




1. $P$ connects two outer vertices $u$ and $v$.
2. $u,v$ is a separation pair of $G$.
3. $P$ lies on an inner face.
4. $P$ does not pass through any outer edge and any outer vertex other than $u$ and $v$.

- *G* is internally 3-connected
- G has no separation pair $\{u,v\}$ s.t. both *u* and *v* are on the same $P_i$ $(1 \le i \le 3)$

➡ *G* has a canonical decomposition

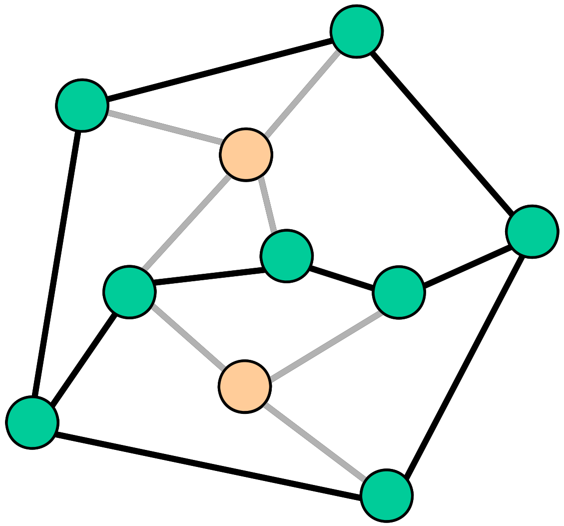



1. *P* connects two outer vertices *u* and *v*.
2. *u*, *v* is a separation pair of *G*.
3. *P* lies on an inner face.
4. *P* does not pass through any outer edge and any outer vertex other than *u* and *v*.

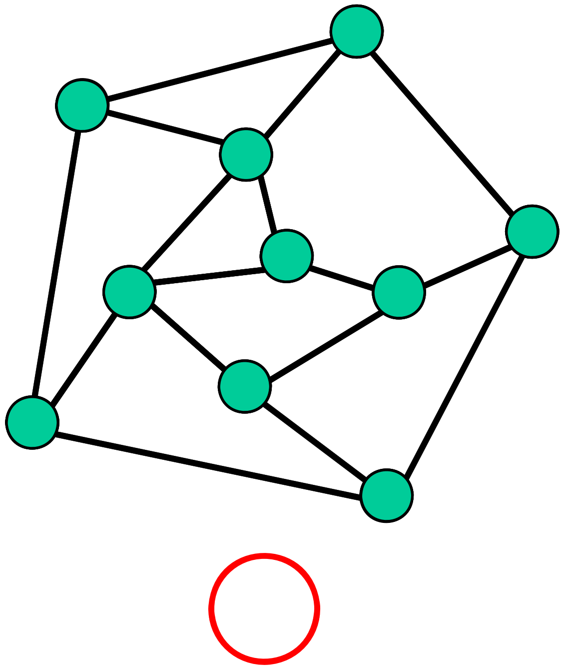
plan of proof :

find a minimal chord-path P, then choose such a vertex set.

# 3-connected plane graph



# 3-connected plane graph



# 3-connected plane graph