

Necessary and Sufficient Condition for a Graph to be Three-Terminal Series-Parallel

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Abstract—A “three-terminal series-parallel graph” is defined to be a three-terminal graph which is constructed by means of repeating only specified series and parallel connections. This definition is based upon our previous work about transformerless interconnections of two-port networks. Some properties of the graph are shown under the foregoing definition and, especially, a theorem is given stating that a three-terminal graph is three-terminal series-parallel if and only if neither of two certain graphs can be obtained from it by opening or shorting some edges.

I. INTRODUCTION

THE PROPERTIES of a two-terminal series-parallel network or graph have been studied by network and graph theorists, in order to apply them to the analysis and synthesis of LCR one-port or a switching network [1]–[6]. It suffices here to remember the Bott–Duffin synthesis of

a positive real function, by which a one-port is realized without use of transformers and is constructed by means of repeating only series and parallel connections [2]. The one-port so obtained is a two-terminal series-parallel network. Another Duffin theorem, well known in graph theory, states that a (two-terminal) graph is (two-terminal) series-parallel if and only if it has no certain graph as a subcontraction [3], where subcontraction of a graph is defined as in the following section.

Three-terminal series-parallel networks [7]–[11] have important theoretical and practical uses, particularly when a transformerless three-terminal network is discussed. Adams has tried to define a three-terminal series-parallel graph [7]. However, his definition seems to be ambiguous from a graph-theoretical standpoint of view. In this paper, an explicit definition for it will first be given, using a previous result [10] on transformerless interconnections between two two-ports. Some consequences follow immediately from the definition. A subsequent theorem, which is the

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main result of this paper, reveals that a three-terminal graph is three-terminal series-parallel if and only if it has neither of the two types of graphs shown in Fig. 5 as a subcontraction.

II. PRELIMINARY DEFINITIONS

Let $G(V,E)$ (or, simply, G) denote a (nonoriented) graph consisting of a finite set V of vertices and a finite set E of edges. Note that in the context of this paper a graph G may have self-loops; also it may have multiple edges between two vertices.

Definition 1

A graph G is called a *two-terminal graph* when two distinct vertices of V are designated as the *terminal vertices* of G . Similarly, when three distinct vertices of V are designated as the terminal vertices of G , G is called a *three-terminal graph*.

Definition 2

A two-terminal graph G is said to be *two-terminal nonseparable*, when the graph obtained by adding an edge between the pair of terminal vertices of G is a nonseparable graph [4]. Similarly, when the graph obtained by adding an edge between each of the respective pairs of the terminal vertices of a three-terminal graph G is a nonseparable graph, G is said to be *three-terminal nonseparable*.

It is noted that all the terminal vertices are distinct from each other, and also that self-loops are looked upon as separable components in a graph. The class of two-terminal nonseparable graphs includes the two-terminal graph consisting of only a couple of isolated vertices, and the class of three-terminal nonseparable graphs includes the three-terminal graph consisting of only three isolated vertices, as the simplest examples. Being a two- or three-terminal nonseparable graph implies being a two- or three-terminal graph, respectively. A two- or three-terminal nonseparable graph represents a physical network in which every element affects the relevant network function(s).

The simple removal of an edge from G is called *opening* that edge. Similarly, *shorting* an edge of G refers to the following operation: the pair of incident vertices of the edge is coalesced (into one vertex) and the edge is removed from the resulting graph.

Definition 3

Suppose E_o and E_s are two edge sets in a graph $G(V,E)$, such that $E_o, E_s \subseteq E$ and $E_o \cap E_s = \phi$. After opening all edges of E_o and shorting all edges of E_s , the resulting graph G_{sb} is called a *subcontraction* of G where isolated vertices produced by the removal of edges are removed.

This original definition is a little modified when G is a three- (or two-) terminal graph, as most of the G 's appearing in this paper are; every terminal vertex of G is preserved for that of G_{sb} ; where an isolated one produced by the removal of edges remains a terminal vertex in G_{sb} ; while two or three of them coalesced into one vertex is looked as one terminal vertex in G_{sb} , and then G_{sb} is a two- (or one-) terminal graph. When a distinction is needed, we

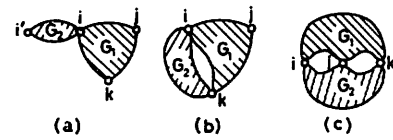


Fig. 1. Basic connections generating three-terminal series-parallel graphs, (a) series connection, (b) parallel (I) connection, and (c) parallel (II) connection.

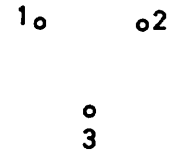


Fig. 2. Basic three-terminal series-parallel graph G_b .

call such G_{sb} a subcontraction of G with preserved terminal vertices, and denote it by $G_{sb} = G[E_o, E_s]$ or simply $G_{sb} < G$.

III. THREE-TERMINAL SERIES-PARALLEL GRAPHS

Our attention is focused on the graph of a two-port network with three terminals which consists of component two-ports interconnected with one another in such a way that the immittance matrix of the composite two-port is given by the sum of appropriate immittance matrices of the component two-ports. Such a two-port occurs, according to Nishizeki [10], only when the component two-ports and the interconnection between them can be represented together by a three-terminal series-parallel graph as defined later. Note that a two-port with four terminals, and one which can not be described by the sum formula, are both outside the interests of this paper.

Let two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ have no vertex in common; i.e., $V_1 \cap V_2 = \phi$. By a *series connection* we mean the operation of coalescing one of the terminal vertices of a two-terminal graph G_2 with one of the terminal vertices of a three-terminal graph G_1 . Fig. 1(a) illustrates a series connection; note that the resultant graph is taken as a three-terminal graph with terminal vertices i' , j , and k . By a *parallel (I) connection* between G_1 and G_2 , we mean the operation of coalescing two of the terminal vertices of a three-terminal graph G_1 with the two terminal vertices of a two-terminal graph G_2 , as illustrated in Fig. 1(b). And by a *parallel (II) connection* between two three-terminal graphs G_1 and G_2 , we mean the operation of coalescing each of the three terminal vertices of G_1 with a distinct terminal vertex of G_2 , as illustrated in Fig. 1(c). In either case, the terminal vertices i, j , and k of G_1 are also the terminal vertices of the composite graph. The parallel (I) and parallel (II) connections are called together *parallel connections* when distinction between them is not required.

Now, we give an inductive definition for a three-terminal series-parallel graph explicitly as follows.

Definition 4

1°—The graph G_b shown in Fig. 2 is a three-terminal series-parallel graph with respect to the terminal vertices 1, 2, and 3.

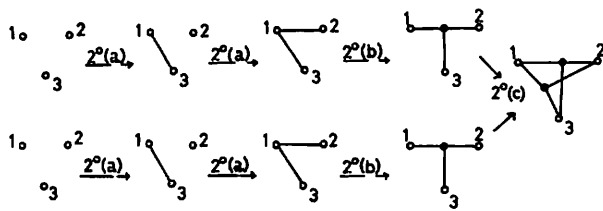


Fig. 3. Generation process for twin- T -type network.

2° —The graph obtained by (a) a parallel (I) connection between a three-terminal series-parallel graph and a two-terminal nonseparable graph, (b) a series connection between a connected three-terminal series-parallel graph and a connected two-terminal nonseparable graph, or (c) a parallel (II) connection between two three-terminal series-parallel graphs, is a three-terminal series-parallel graph with respect to its terminal vertices.

For short, a graph G is said to be *three-terminal series-parallel* when G is a three-terminal series-parallel graph with respect to its own terminal vertices, unless otherwise specified. Being a three-terminal series-parallel graph implies being a three-terminal graph. It should be remarked in Definition 4 that the class of three-terminal series-parallel graphs includes G_b , that consists of isolated vertices only.

There are in practical use many two-ports which are represented simply by three-terminal series-parallel graphs, such as the T , π , ladder, twin- T and bridged- T types. For example, one possible way to generate the graph of the twin- T two-port is shown in Fig. 3. It is noted that the process is not always unique.

Some fundamental properties of the three-terminal series-parallel graph follow immediately from Definition 4.

Property 1: A three-terminal series-parallel graph is three-terminal nonseparable, and, therefore, has no self-loop, but is not necessarily connected.

Property 2: A three-terminal series-parallel graph remains three-terminal series-parallel, even if the order of designations on the terminal vertices is arbitrarily changed in the graph.

Property 3: A three-terminal series-parallel graph is not always two-terminal series-parallel with respect to an arbitrarily prescribed pair of terminal vertices.

Property 4: The impedance or admittance matrix describing a two-port represented by a three-terminal series-parallel graph can always be derived by repeating the summation of impedance or admittance matrices, appropriately.

IV. NECESSARY AND SUFFICIENT CONDITION FOR A GRAPH TO BE THREE-TERMINAL SERIES-PARALLEL

To establish the necessary and sufficient conditions for a three-terminal graph to be three-terminal series-parallel, two forbidden subcontractions will be introduced. The following lemma is first shown.

Lemma 1

A subcontraction of a three-terminal series-parallel graph with preserved terminal vertices is three-terminal series-parallel if it is three-terminal nonseparable.

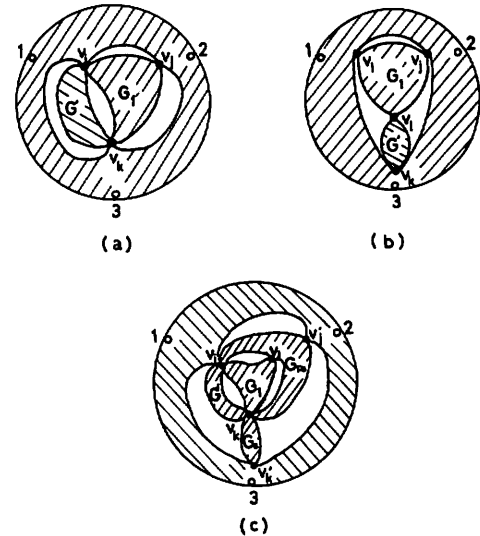


Fig. 4. Three-terminal series-parallel graph G , where G' is connected (a) in parallel (I) or (b) in series with G_1 . Few connections following from the parallel (I) connection of G' in (a) are illustrated in (c).

Proof: Let $G(V, E)$ be an arbitrary three-terminal series-parallel graph with respect to the terminal vertices 1, 2, and 3, and assume a subcontraction G_{sb} of G to be a three-terminal nonseparable graph, where $G_{sb} = G[E_o, E_s]$, $E_o, E_s \subseteq E$ and $E_o \cap E_s = \phi$. Note E_s does not involve such edges that G_{sb} becomes two-terminal by shorting them, because G_{sb} must be three-terminal. Set $E_{os} = E_o \cup E_s$. As G is three-terminal series-parallel, G can be generated from the original G_b shown in Fig. 2 through the application of the connections mentioned in 2° of Definition 4. Hence, any edge in E belongs to a certain two-terminal nonseparable graph $G'(V', E')$, where $V' \subseteq V$ and $E' \subseteq E$, which occurs on a step of the generation process of G . Now suppose G' to contain at least one edge of E_{os} . It may also be supposed, without loss of generality, that the generation process is divided into three main parts, ordered as follows: ① G_b becomes G_1 after repeating the connections of 2° (a), (b), (c) appropriately, ② G_1 becomes G_2 with G' connected as in either 2° (a) or (b), and then ③ G_2 finally becomes G on repeating the connections 2° (a), (b), (c) appropriately.

Assume that $G'' \equiv G'[E' \cap E_o, E' \cap E_s]$, and $G^{(1)} \equiv G[E' \cap E_o, E' \cap E_s]$. Obviously, $G_{sb} < G^{(1)}$, and G'' is a subgraph of $G^{(1)}$. Here, G'' is two-terminal nonseparable or has a vertex only. If not so, $G^{(1)}$ is three-terminal separable (as seen in the graph shown in Fig. 4(a) and (b) where G' 's are replaced by G'' 's), and therefore, $G_{sb} (= G^{(1)}[(E - E') \cap E_o, (E - E') \cap E_s])$ is so, contrary to the assumption. From this it follows that G'' consists of, either i) some edges, ii) one vertex only, or iii) two vertices only. In all these cases, it will first be shown that $G^{(1)}$ or a certain subcontraction $G^{(2)}$ of $G^{(1)}$ will be three-terminal series-parallel.

i) *When G'' consists of at least one edge:* Making use of G'' instead of G' in part ② of the generation process of G yields $G^{(1)}$ instead of G . Hence, $G^{(1)}$ is three-terminal series-parallel.

ii) *When G'' consists of a vertex only:* Let us first consider the case where 2° (b) is used in part ②, as shown in Fig. 4(b). Then, $G^{(1)}$ is evidently three-terminal series-

parallel, because $G^{(1)}$ is obtained instead of G if part ② is skipped in the generation process of G . (Notice G_1 must be connected, otherwise the series connection of G' is not applicable to G_1 , according to Definition 4, 2°(b).) We next consider the case where 2°(a) is used in part ② as shown in Fig. 4(a). The case when G is disconnected is trivial. Suppose G to be connected. It may also be supposed, without loss of generality, that G_2 (consisting of G_1 and G') becomes G_3 with G_{ps} after repeating 2°(a), (b), (c), G_3 becomes G_4 with a connected two-terminal nonseparable graph G_5 which is connected with v_k of G_3 by series connection, and then G_4 becomes G on repeating 2°(a), (b), (c) in part 3 of the generation process of G , as illustrated in Fig. 4(c). Here, the terminal vertices of G_3 are v_i, v_j' , and v_k , and those of G_4 are v_i, v_j' , and v_k' . G_{ps} may not exist. G_5 must exist in G , since, if not, $G_{sb} (< G^{(1)})$ may be two-terminal, contrary to the assumption. G_3 (consisting of G_1, G' , and G_{ps}) is connected, according to Definition 4, 2°(b). If $G^{(1)}$ for such G is three-terminal nonseparable, (in other words, if no separable component first appears when G' is subcontracted into one vertex in order to produce $G^{(1)}$), then $G_3[E' \cap E_o, E' \cap E_s]$ (consisting of G_1 and G_{ps} with v_i and v_j coalesced) goes to a connected two-terminal nonseparable graph with respect to v_i and v_j' , and, hence, $G_4[E' \cap E_o, E' \cap E_s]$ (consisting of G_3, G_{ps} and G_1 with v_i and v_k coalesced) is a connected three-terminal series-parallel graph (with respect to v_i, v_j' , and v_k'). Furthermore, $G^{(1)}$ is obviously three-terminal series-parallel, since $G^{(1)}$ is obtained instead of G if $G_4[E' \cap E_o, E' \cap E_s]$ is used instead of G_4 (consisting of G_3, G_{ps} , and G_1) in the generation process of G . Next, if $G^{(1)}$ has separable components, then the edge set E_{se} of such components must be such that $E_{se} \subseteq E_{os}$, because $G_{sb} < G^{(1)}$ and G_{sb} is three-terminal nonseparable by the assumption. And any edge of E_{se} is only in G_1 or G_{ps} as seen in Fig. 4(c). Therefore, the graph $G^{(2)} = G^{(1)}[E_{se} \cap E_o, E_{se} \cap E_s]$ is surely three-terminal nonseparable (with respect to the terminal vertices 1, 2, and 3). Moreover, $G^{(2)}$ is three-terminal series-parallel, since $G^{(2)}$ is obtained instead of G if the connected three-terminal series-parallel graph $G_4[(E' \cup E_{se}) \cap E_o, (E' \cup E_{se}) \cap E_s]$ is used instead of G_4 in the generation process of G .

iii) When G'' consists of two vertices only: In similar fashion as above, it can be shown that $G^{(1)}$ or $G^{(2)}$ exists and is three-terminal series-parallel (with respect to vertices 1, 2, and 3).

Each of the edge sets of $G^{(1)}$ and $G^{(2)}$ is properly included in E and, moreover, $G_{sb} < G^{(1)}, G^{(2)}$. Hence, after iteration of the foregoing reductions for all edges of E_{os} , $G^{(1)}$ or $G^{(2)}$ becomes G_{sb} ; that is, G_{sb} is three-terminal series-parallel.

Q.E.D.

The following is a well-known lemma on nonseparable graphs.

Lemma 2

A connected graph with at least three vertices is nonseparable if and only if every two vertices of the graph lie on a common cycle [12].

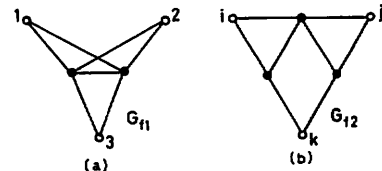


Fig. 5. Minimal three-terminal non-series-parallel graphs (a) G_{f1} and (b) G_{f2} , where $(i, j, k) = (1, 2, 3)$.

Verification for the following theorem comes from applications of Lemmas 1 and 2. Some terms are defined beforehand, for convenience. By the *series removal* of G_2 from G , we mean the removal of a two-terminal graph G_2 from a three-terminal graph G , which leaves the resulting three-terminal graph G_1 with the new terminal vertex i , as illustrated in Fig. 1(a). By the *parallel (I) or parallel (II) removal* of G_2 from G , we mean the removal of a two-terminal graph or a three-terminal graph G_2 from a three-terminal graph G , which leaves the resulting three-terminal graph G_1 , as illustrated in Fig. 1(b) or (c), respectively. In these cases, the terminal vertices in G_1 are also i, j , and k which are those in G . Parallel (I) and parallel (II) removals are together called, simply, *parallel removals*.

Theorem

A three-terminal nonseparable graph G is three-terminal series-parallel if and only if G has neither G_{f1} nor G_{f2} , shown in Fig. 5, as a subcontraction of G with preserved terminal vertices.

Proof

Necessity: It is obvious that G_{f1} and G_{f2} are not three-terminal series-parallel, though they are three-terminal nonseparable. However, according to Lemma 1, a subcontraction G_{sb} of G with preserved terminal vertices must be three-terminal series-parallel if it is three-terminal nonseparable. Therefore, neither $G_{f1} < G$ nor $G_{f2} < G$.

Sufficiency: Assume a three-terminal non-series-parallel graph G to be three-terminal nonseparable. G is connected, since if G is disconnected and three-terminal nonseparable, G must be three-terminal series-parallel, contrary to the hypothesis. Apply to G the series and parallel removals repeated as far as possible, where every graph removed through the removals is to be a connected two- or three-terminal nonseparable graph, according to Definition 4. And, we denote the resulting three-terminal graph by $G^{(1)}$. Note that it is always possible to choose the three-terminal graph remaining after each of the above removals to be three-terminal nonseparable and three-terminal non-series-parallel, since if it is impossible G is obviously three-terminal separable or three-terminal series-parallel, contrary to the assumption. Therefore, we can suppose that $G^{(1)}$ is also three-terminal nonseparable and three-terminal non-series-parallel. Furthermore, $G^{(1)}$ is connected because parallel (I) removal is not applicable to $G^{(1)}$ any more. $G^{(1)}$ is nonseparable, too. If $G^{(1)}$ is separable, then $G^{(1)}$ is three-terminal series-parallel, or applicable of further series removal, so that contradictions are caused. Thus $G^{(1)}$ is a connected and nonseparable graph containing more than two vertices. There-

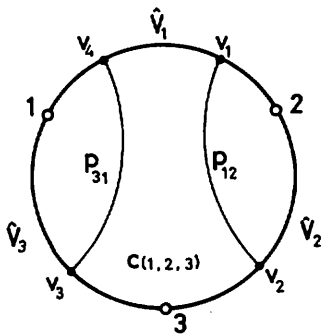


Fig. 6. A graph consisting of $c(1,2,3)$, p_{12} and p_{31} in case (1).

fore, every two vertices in $G^{(1)}$ must belong to a common cycle according to Lemma 2.

Let us consider the case when there exists a cycle $c(1,2,3)$ passing through all the terminal vertices 1, 2, and 3 in $G^{(1)}$, and consider the other case later.

(1) When $c(1,2,3)$ exists: Write $c(1,2,3)$ by ordered multiplication of vertices as follows:

$$c(1,2,3) = 1 \cdot \hat{V}_1 \cdot 2 \cdot \hat{V}_2 \cdot 3 \cdot \hat{V}_3 \cdot 1.$$

Let V_i ($i = 1,2,3$) denote the set of vertices appearing in \hat{V}_i , then neither of V_i 's is empty since $G^{(1)}$ contains no edge incident to two terminal vertices. P_{12} (P_{23} or P_{31}) denotes the set of paths in $G^{(1)}$, each of which begins with a vertex in V_1 (V_2 or V_3), ends with a vertex in V_2 (V_3 or V_1), and passes through no vertex on $c(1,2,3)$ except the beginning and ending vertices of that path. Then, at least two of the three path sets are not empty, for otherwise parallel removal would be still applicable to $G^{(1)}$. Thus, without loss of generality we can assume, by Property 2, that P_{12} and P_{31} are not empty. Let $p_{12}(v_1, v_2)$ or $p_{31}(v_3, v_4)$ be a path in P_{12} or P_{31} , respectively. Here, $p(i, j)$ designates a path which begins with a vertex i and ends with a vertex j . When p_{31} does not meet any vertex on p_{12} , $G^{(1)}$ goes to the graph depicted in Fig. 6 by opening all edges except those belonging to $c(1,2,3)$, p_{12} and p_{31} . Furthermore, the graph can be ultimately modified into G_{f2} of Fig. 5(b) by shorting some appropriate edges. Therefore, $G_{f2} < G^{(1)}$. Also, when p_{31} meets one or more vertices on p_{12} , it can be similarly shown that $G_{f2} < G^{(1)}$.

(2) When no $c(1,2,3)$ exists: Since $G^{(1)}$ is connected and nonseparable it contains two cycles, $c(1,2,3)$ passing through 1 and 2, but not 3, and $c(1,2,3)$ passing through 1 and 3, but not 2. Now, we write them in terms of the sum of paths or the ordered multiplication of vertices as follows:

$$c(1,2,3) = p_1(1,2) + p_2(2,1)$$

$$= 1 \cdot \hat{V}_1 \cdot 2 \cdot \hat{V}_2 \cdot 1$$

$$c(1,2,3) = p_3(1,3) + p_4(3,1)$$

$$= 1 \cdot \hat{V}_3 \cdot 3 \cdot \hat{V}_4 \cdot 1.$$

Let V_i ($i = 1,2,3,4$) be the set of vertices appearing in \hat{V}_i , then neither of the V_i 's is empty. Opening all edges incident to the terminal vertices of $G^{(1)}$, and removing all the terminal vertices from the resulting graph, yields a connected sub-

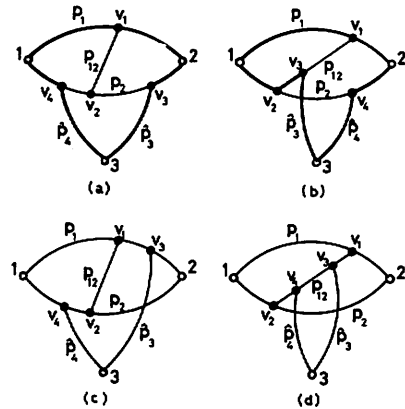


Fig. 7. The locations of v_3 and v_4 in the graph $G^{(1)}$ in (a) case 1, (b) case 2, (c) case 3, and (d) case 4.

graph $G^{(2)}$. If $G^{(2)}$ is not connected, $G^{(1)}$ contradicts the assumption, because such $G^{(1)}$ is still applicable of the parallel removal. There are now three cases to consider, depending on whether p_3 and p_4 meet $c(1,2,3)$ or not.

(2-1) When each p_3 and p_4 meets $c(1,2,3)$ at one or more vertices except the terminal vertex 1: Since $G^{(2)}$ is connected, there exists such a path $p_{12}(v_1, v_2)$ in $G^{(1)}$ that begins with a vertex $v_1 \in V_1$, ends with a vertex $v_2 \in V_2$, passes through no vertex on $c(1,2,3)$ except v_1 and v_2 , and passes through no terminal vertex. Let v_3 and v_4 be the first vertices at which $p_3(3,1)$ and $p_4(3,1)$ meet $c(1,2,3)$ or p_{12} , respectively. It is obvious from the assumption that v_3 and v_4 are distinct from each other and identical neither with 1 nor with 2. It suffices to consider the following four cases, due to the symmetry between the pair of p_1 and p_2 and the pair of p_3 and p_4 in regard to locations of v_3 and v_4 :

- 1) v_3 and v_4 are both on p_2 ,
- 2) v_3 is on p_{12} except v_1 , and v_4 is on p_2 except v_2 ,
- 3) v_3 is on p_1 except 1 and 2, and v_4 is on p_2 except 1 and 2, and
- 4) v_3 and v_4 are both on p_{12} .

Let \hat{p}_3 denote the portion of p_3 between v_3 and 3, and \hat{p}_4 the portion of p_4 between v_4 and 3. Fig. 7 shows the scheme of p 's, \hat{p} 's and v 's in $G^{(1)}$ for the above four cases. By observing the traces of the bold lines in Fig. 7(a) and (b), we find $c(1,2,3)$'s, $1 \cdot \hat{V}_1 \cdot 2 \cdots v_3 \cdots 3 \cdots v_4 \cdots 1$ and $1 \cdot \hat{V}_1 \cdot 2 \cdots v_4 \cdots 3 \cdots v_3 \cdots v_2 \cdots 1$, in the respective $G^{(1)}$'s. Thus cases 1) and 2) reduce to the preceding case (1). In cases 3) and 4), it follows that $G_{f1} < G^{(1)}$ since the graphs shown in Fig. 7(c) and (d) are subgraphs of $G^{(1)}$, and G_{f1} can be obtained from these by shorting some edges.

(2-2) When p_3 meets $c(1,2,3)$ at one or more vertices except the terminal vertex 1, and p_4 does only at 1: Let v_3 be the first vertex at which $p_3(3,1)$ meets $c(1,2,3)$. We can assume that $v_3 \in V_2$, without loss of generality. Refer to the scheme of p 's and v 's in $G^{(1)}$, as shown in Fig. 8(a), and we can observe $c(1,2,3) = 1 \cdot \hat{V}_1 \cdot 2 \cdots v_3 \cdots 3 \cdot \hat{V}_4 \cdot 1$ on the trace of bold lines. Thus this case is reduced to the preceding case (1).

(2-3) When both p_3 and p_4 meet $c(1,2,3)$ only at the terminal vertex 1: Since $G^{(2)}$ is connected, we can assume

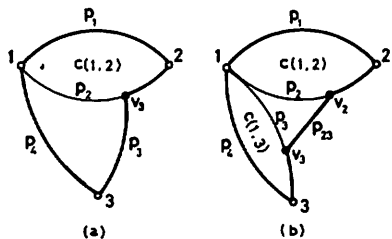


Fig. 8. The subgraphs of $G^{(1)}$ in (a) case (2-2) and (b) case (2-3).

without loss of generality that there exists a path p_{23} which begins with a vertex $v_2 \in V_2$, ends with a vertex $v_3 \in V_3$, and passes through no vertex on $c(1,2,3)$ and $c(1,2,3)$ except v_2 and v_3 . Then, as seen in Fig. 8(b), $c(1,2,3) = 1 \cdot \bar{p}_1 \cdot 2 \cdots v_2 \cdots v_3 \cdots 3 \cdot \bar{p}_4 \cdot 1$ exists in $G^{(1)}$. Therefore, this case is also reduced to the preceding case (1).

As above, it is evidently shown that $G^{(1)} > G_{f1}$ or G_{f2} . Since $G > G^{(1)}$, it is concluded $G > G_{f1}$ or G_{f2} . Thus the theorem is established. Q.E.D.

A three-terminal non-series-parallel graph is allowed to have both G_{f1} and G_{f2} as its subcontractions with preserved terminal vertices, simultaneously, as seen in Fig. 7(b).

V. CONCLUSIONS AND REMARKS

A new criterion of a three-terminal series-parallel graph has been exhibited in terms of the forbidden subcontractions of the graph. Just as the well-known Kuratowski graphs, K_5 and $K_{3,3}$, characterize a nonplanar graph, the forbidden subgraphs or subcontractions play an important role in many aspects of graph theory [13]. It is also of great interest that the Kuratowski graph K_5 is derived by adding an edge between each pair of the terminal vertices of G_{f1} of Fig. 5(a).

A three-terminal series-parallel graph is not necessarily planar, while a two-terminal series-parallel one is always planar [3]. This can be immediately recognized from the fact that a two-terminal nonseparable graph used in constructing a three-terminal series-parallel graph in Definition 4 need not always be planar. Even if only two-terminal nonseparable and planar graphs are used in Definition 4, an obtainable three-terminal series-parallel graph is not always planar. This is deduced from the fact the graph obtained by parallel connecting three T-type graphs is the other Kuratowski graph $K_{3,3}$.

It may not be an easy matter to decide in practice whether a given large and complex graph is three-terminal series-parallel or not. An algorithm for making such a decision easy will be useful. Such an algorithm may perhaps be composed of steps corresponding to the steps appearing in the proof of the above theorem.

A similar theorem to that above holds true for a three-terminal series-parallel-cascade graph, which is defined as a three-terminal graph obtained by repeating series, parallel and cascade connections appropriately [14]. Moreover, these theorems can also be represented in terms of a "homeomorphism" instead of a "subcontraction," as is the Kuratowski theorem [12].

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