Linear-Time Computability of Combinatorial Problems on Series-Parallel Graphs

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Abstract. A series-parallel graph can be constructed from a certain graph by recursively applying "series" and "parallel" connections. The class of such graphs, which is a well-known model of series-parallel electrical networks, is a subclass of planar graphs. It is shown in a unified manner that there exist linear-time algorithms for many combinatorial problems if an input graph is restricted to the class of series-parallel graphs. These include (i) the decision problem with respect to a property characterized by a finite number of forbidden graphs, (ii) the minimum edge (vertex) deletion problem with respect to the same property as above, and (iii) the generalized matching problem. Consequently, the following problems, among others, prove to be linear-time computable for the class of series-parallel graphs: (1) the minimum vertex cover problem, (2) the maximum outerplanar (induced) subgraph problem, (3) the minimum feedback vertex set problem, (4) the maximum (induced) line-subgraph problem, (5) the maximum matching problem, and (6) the maximum disjoint triangle problem.

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1. *Introduction*

A large number of combinatorial problems defined on graphs are NP-complete, and hence there is probably no polynomial-time algorithm for any of them [1]. A number of such problems can be formulated as a "minimum edge (vertex) deletion problem" with respect to some graph property Q [14, 15, 23]. The problem asks a minimum number of edges (vertices) of a given graph whose deletion results in a graph satisfying Q. Various other problems can be formulated as a "generalized matching problem" in which one would like to find a maximum number of vertex-disjoint copies of a fixed graph B contained in an input graph [13]. It should be noted that a graph property Q can be often characterized by (possibly an infinite number of) "forbidden (induced) subgraphs," that is, a graph G satisfies Q if and only if G contains none of the forbidden graphs as an (induced) subgraph [3, 11, 15].

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Some of the combinatorial problems which are NP-complete for general graphs remain so even for a restricted class of graphs [7]. However, it has been shown by ad hoc methods that polynomial-time algorithms are available for some combinatorial problems on various special classes of graphs, such as planar graphs, regular graphs, bipartite graphs, or series–parallel graphs [8].

In this paper we consider a special class of graphs, called series–parallel graphs, which can be obtained from a certain graph by recursively applying series and parallel connections. The class of such graphs, which is a well known model of series–parallel electrical networks, is a restricted class of planar graphs. Many practical problems defined on such graphs can be efficiently solved, for example, resistance of electrical networks, reliability of systems, and scheduling [4, 16, 17]. The following question naturally arises: Do there exist polynomial-bounded algorithms for *all* combinatorial problems defined on such a class of graphs? One can easily see that not every combinatorial problem is polynomial-time computable even if restricted to series–parallel graphs. However, we show in a unified manner that a number of combinatorial problems are linear-time computable for series–parallel graphs. Such a rather broad class of problems includes:

(i) the decision (i.e., yes–no) problem with respect to any property **Q** characterized by a finite number of forbidden (induced or homeomorphic) subgraphs, in which one would like to decide whether an input graph satisfies **Q**;
(ii) the minimum edge (vertex) deletion problem with respect to the same property as above; and
(iii) the generalized matching problem.

Hence the following problems, among others, prove to be linear-time computable for the class of series–parallel graphs: (1) the minimum vertex cover problem, (2) the maximum outerplanar (induced) subgraph problem, (3) the minimum feedback vertex set problem, (4) the maximum line-subgraph problem, (5) the maximum matching problem, and (6) the maximum disjoint triangle problem. Some of these problems have individually been shown to be polynomial-time computable for the class of series–parallel graphs or some larger class containing all such graphs [2, 5, 8, 22].

2. *Preliminary*

Most of the graph-theoretical terms used in this paper are standard (see, e.g., [9]). We therefore limit ourselves to defining the most commonly used terms and those that may produce confusion.

A *multigraph* $G = (V, E)$ consists of a finite set V of *vertices* and a finite multiset E of *edges*, each of which is a pair of distinct vertices. Throughout this paper we simply call them *graphs*, since we consider only multigraphs. A graph $G' = (V', E')$ is a *subgraph* of another $G = (V, E)$ if V' is a subset of V and E' is a subset of E. If $V' = V$, G' is denoted by $G' = G - E_s$ where $E_s = E - E'$. We write $G' \subset G$ if G' is a subgraph of G. For any subset W of the vertices of a graph G, the *induced subgraph* on W is the maximal subgraph of G with vertex set W and is denoted by $G - V_s$, where $V_s = V - W$.

Two edges of a graph are *series* if they are incident to a vertex of degree 2 and are *parallel* if they join the same pair of distinct vertices. A *series–parallel graph* is defined recursively as follows [4].

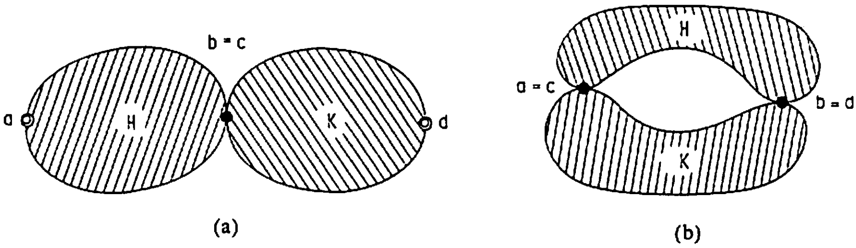


FIG. 1. (a) Series connection of type I. (b) Parallel connection.

Definition 1. A graph consisting of two vertices joined by two parallel edges is series–parallel. If G is a series–parallel graph, then a graph obtained from G by replacing any edge of G by series or parallel edges is series–parallel.

Definition 2. A graph $G = (V, E)$ is called a *two-terminal graph* when two distinct vertices of V are distinguished from the other vertices and also from each other; the two distinct vertices are called the *first terminal* and the *second terminal*, respectively. We write $G = (V, E, x, y)$ if x is the first terminal of G and y the second.

A two-terminal graph may have one or two *virtual terminals*, denoted by v_{ti} ($i =$ 1, 2), which are necessarily isolated and distinguished from a usual terminal. They play a special role in the connections or separations of the succeeding sections. When we wish to discriminate a nonvirtual vertex from virtual one, we call it a *real vertex*. An *underlying graph* of a two-terminal graph G is a graph consisting of all edges and all real vertices of G.

We will introduce a two-terminal series–parallel graph, which is slightly different from a series–parallel graph. We first define two kinds of connections of two-terminal graphs, which are used to construct two-terminal series–parallel graphs, as follows.

Definition 3. Let $H = (V_H, E_H, a, b)$ and $K = (V_K, E_K, c, d)$ be two two-terminal graphs having no vertex in common: $V_H \cap V_K = \emptyset$.

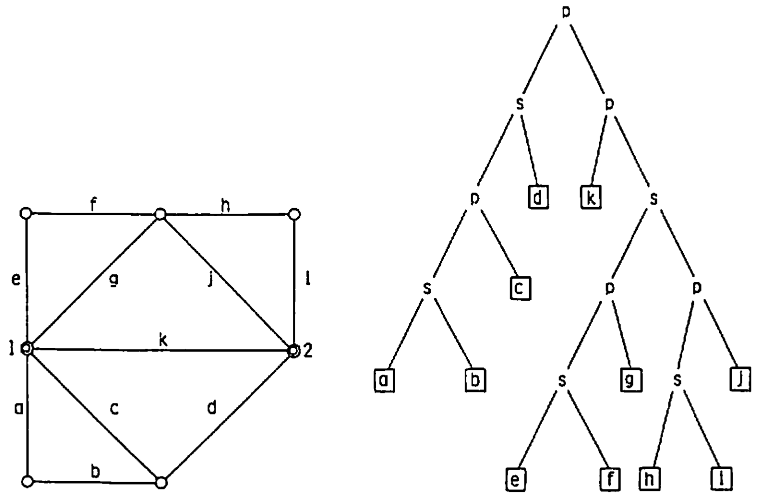
(a) H and K are *series connectable* (*in type I*) if the second terminal b of H and the first terminal c of K are both real. By a *series connection* (*of type I*) of two-terminal graphs H and K, we mean the two-terminal graph G_s obtained from H and K by identifying b of H with c of K as illustrated in Figure 1a. Note that the resultant graph G_s is regarded as a two-terminal graph with terminals a and d.
(b) H and K are *parallel connectable* if both a and c are either real or virtual and both b and d are also either real or virtual. By a *parallel connection* of H and K we mean the two-terminal graph G_p obtained from H and K by identifying a with c and b with d. Figure 1b illustrates a parallel connection; the terminal vertices of H are also the terminal vertices of the composite graph.

If a two-terminal graph G_s (or G_p) is obtained from two two-terminal graphs H and K by a series connection of type I (or parallel connection), we write $G_s = H * K$ (or $G_p = H//K$) and say that H and K are *series separations of type I* of G_s (or *parallel separations* of G_p).

A *two-terminal series–parallel* (*TTSP*) *graph* is recursively defined as follows.

Definition 4 (*Two-Terminal Series–Parallel Graphs*)

(i) A two-terminal graph consisting of two vertices joined by a single edge is TTSP and is called a *minimum series–parallel graph*, denoted by $G_{\min}$.



(a) (b)

FIG. 2. (a) A TTSP graph with terminals 1 and 2. (b) Its binary decomposition tree.

(ii) If H and K are TTSP graphs, then the two-terminal graphs $H * K$ and $H//K$ are TTSP.

It is known that every series–parallel graph is TTSP if one designates appropriate vertices, for example, two ends of an edge, as the terminals [4]. Figure 2 illustrates a TTSP graph and its binary decomposition tree, in which node label s or p indicates series or parallel connection, respectively.

3. *Decision Problems on TTSP Graphs*

In this section we consider a decision (i.e., yes–no) problem with respect to property Q for TTSP graphs, in which one would like to determine whether a given TTSP graph satisfies Q or not. We show that whenever Q can be characterized by a finite number of forbidden (induced) subgraphs, the decision problem is linear-time computable.

Throughout this paper we often denote by Q itself the *set* of all (two-terminal) graphs satisfying *property* Q, so that $G \in \mathbf{Q}$ if and only if G satisfies property Q. We also denote "and" by $\wedge$ and "or" by $\vee$. On the basis of the recursive definition of a TTSP graph we employ "divide and conquer" to solve our problems on TTSP graphs. The requirements for the success and efficiency of the method are formulated as follows.

LEMMA 1. *Let* Q *be a property defined on two-terminal graphs. The decision problem with respect to* Q *is linear-time computable for TTSP graphs if there exists a finite set of properties* $\mathbf{P} = \{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_k\}$ *such that*

(*i*) $\mathbf{Q} \in \mathbf{P}$;
(*ii*) *it is known whether* $G_{min} \in \mathbf{Q}_r$ *or not for each property* $\mathbf{Q}_r \in \mathbf{P}$; *and*
(*iii*) *for each property* $\mathbf{Q}_r \in \mathbf{P}$, *there exist two sets of properties* $\{\mathbf{Q}_{h1}, \mathbf{Q}_{h2}, \ldots, \mathbf{Q}_{ht}\} \subset \mathbf{P}$ *and* $\{\mathbf{Q}_{k1}, \mathbf{Q}_{k2}, \ldots, \mathbf{Q}_{kt}\} \subset \mathbf{P}$ *such that for any two-terminal graph* G_T *satisfying* $G_T = H * K$ (*or* $H//K$),

$$G_T \in \mathbf{Q}_r \quad \textit{iff} \quad \bigvee_{i=1}^{t} (H \in \mathbf{Q}_{hi} \wedge K \in \mathbf{Q}_{ki}),$$

procedure TEST(G):
if $G = G_{\min}$ **then** determine $G_{\min} \in Q_r$ for each $Q_r \in P$
else if $G = H * K$ for H and K both having fewer edges than G
  then TEST(H) and TEST(K), and determine $G \in Q_r$ for each $Q_r \in P$ by using the solutions to H and K and condition (iii) of Lemma 1
  else let $G = H//K$ for H and K both having fewer edges than G;
   TEST(H), TEST(K), and determine $G \in Q_r$ for each $Q_r \in P$ by using the solutions to H and K and condition (iii) of Lemma 1

FIG. 3. A decision algorithm.

where $*$ denotes a series connection of type I and H and K are both two-terminal graphs. (It should be noted that different sets of properties would be used for series and for parallel connections.)

PROOF. Let $G = (V, E, x, y)$ be a given TTSP graph. Consider an extended decision problem in which one would like to determine for *every* $Q_r \in P$ whether $G \in Q_r$ or not. Since $Q \in P$ by (i), the new problem includes the original one. Therefore it is sufficient to verify that the extended decision problem is linear-time computable. We will show that the recursive algorithm shown in Figure 3 solves this problem in time linear in the number of edges of G.

First it should be noted that any TTSP graph G with $e = |E|$ edges can be constructed from e copies of the minimum series–parallel graph $G_{\min}$ by a sequence of series and parallel connections, and that such a sequence can be determined by constructing a binary decomposition tree of G in $O(e)$ time [10, 19, 21]. That is, the total amount of time required for series and parallel separations (i.e., the cost of "divisions") is $O(e)$. Hence we shall verify that the other operations (i.e., "combinations" and "solving the problems on $G_{\min}$'s") require at most $O(e)$ time. Let $T(e)$ denote the total amount of time required for these operations to solve the extended decision problem on G with e edges. By induction on e we prove that $T(e) \leq c_1 e - c_2$ for some constants c_1 and c_2. If $e = 1$, that is, $G = G_{\min}$, then condition (ii) implies that one can determine in a constant time whether $G \in Q_r$ for $r = 1, 2, \ldots, k$. Thus $T(1) \leq c_1 \cdot 1 - c_2$ for appropriate constants c_1 and c_2. If $e \geq 2$, then $G = H * K$ or $H//K$ for two TTSP graphs H and K, both with fewer edges than G. Condition (iii) implies that the solution to G can be obtained by combining the solutions to H and K in a constant time. Note that both k and t are constants independent of the size e of the problem instance. Thus, if H has e_H edges and K e_K edges, then

$$T(e) \leq T(e_H) + T(e_K) + kt,$$

where $e_H, e_K \geq 1$ and $e = e_H + e_K$. The inductive hypothesis implies that

$$T(e_H) \leq c_1 e_H - c_2 \quad \text{and} \quad T(e_K) \leq c_1 e_K - c_2.$$

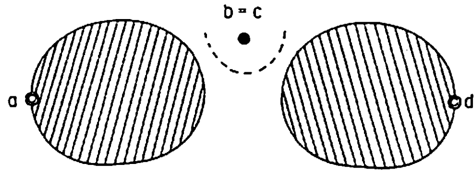
Therefore, by appropriately selecting c_1 and c_2, we have $T(e) \leq c_1 e - c_2$, which completes the proof. □

We need some more definitions.

Definition 5. Let $G = (V, E)$ be a graph, and let x and y be any distinct vertices in V. A two-terminal graph G_T is a *terminal-attached graph* of G if G_T is one of the following:

(i) $G_T = (V, E, x, y)$,
(ii) $G_T = (V \cup \{v_{t1}\}, E, v_{t1}, y)$,

FIG. 4. Series connection of type II.



(iii) $G_T = (V \cup \{v_{t2}\}, E, x, v_{t2})$, and
(iv) $G_T = (V \cup \{v_{t1}, v_{t2}\}, E, v_{t1}, v_{t2})$,

where v_{t1} and v_{t2} are virtual vertices.

Let $G_T = (V, E, x, y)$ and $G'_T = (V', E', x', y')$ be two-terminal graphs. G'_T is a *two-terminal subgraph* of G_T if (i) the underlying graph of G'_T is a subgraph of the underlying graph of G_T, (ii) $x' = x$ if $x \in V'$, otherwise x' is an isolated virtual terminal v_{t1}, and (iii) $y' = y$ if $y \in V'$, otherwise y' is an isolated virtual terminal v_{t2}. If $V' = V$, then we write $G'_T = G_T - E_s$ with $E_s = E - E'$. If E' contains all the edges of E with both ends in V', G'_T is a *two-terminal induced subgraph* of G_T. If $V_s \subset V$, then $G_T - V_s$ denotes the induced subgraph G'_T with vertex set V' consisting of all vertices in $V - V_s$ together with v_{ti} ($i = 1$ or 2) if x (or y) $\in V_s$. We write again $G'_T \subset G_T$ if G'_T is a two-terminal subgraph of G_T.

Definition 6. Let $H = (V_H, E_H, a, b)$ and $K = (V_K, E_K, c, d)$ be two two-terminal graphs having no vertex in common: $V_H \cap V_K = \emptyset$. H and K are *series connectable in type II* if both b and c are virtual. By a *series connection of type II*, we mean the two-terminal graph G_s obtained from the union of H and K by deleting b and c, as illustrated in Figure 4. That is,

$$G_s = (V_H \cup V_K - \{b, c\}, E_H + E_K, a, d).$$

H and K are called *series separations of type II* of G_s.

If G_s is a series connection of H and K, then we write, from now on, $G_s = H * K$ regardless of whether it is type I or II.

Many kinds of graph properties **Q** can be characterized by forbidden (induced) subgraphs [3]. That is, the set of graphs satisfying **Q** can be defined by a set of graphs which they do not contain as (induced) subgraphs. We will show that if the set of forbidden graphs is finite, then the decision problem with respect to such a property **Q** is linear-time computable for series–parallel graphs.

Suppose that property **Q** is defined on graphs by a finite set of forbidden subgraphs $\mathbf{B} = \{B_1, B_2, \ldots, B_q\}$. Thus $G \in \mathbf{Q}$ if and only if G contains no members of **B** as a subgraph. Since the algorithm in Lemma 1 works only on TTSP graphs, we shall first reduce the decision problem defined on graphs to one on two-terminal graphs. We define a set $\mathbf{B}_T$ of forbidden two-terminal subgraphs as the set of all terminal-attached graphs of graphs in **B**. It should be noted that the set $\mathbf{B}_T$ is finite, since the set **B** is finite. Associated with property **Q** on graphs, we define property $\mathbf{Q}_T$ on two-terminal graphs by $\mathbf{B}_T$: $\mathbf{Q}_T$ is the set of all the two-terminal graphs that contain no member of $\mathbf{B}_T$ as a two-terminal subgraph.

LEMMA 2. *Let G_T be any terminal-attached graph of a graph G. Then $G \in \mathbf{Q}$ if and only if $G_T \in \mathbf{Q}_T$.*

PROOF. Suppose $G_T \notin \mathbf{Q}_T$. Then there exists $B'_i \in \mathbf{B}_T$ such that $B'_i \subset G_T$. Let $B_i \in \mathbf{B}$ be the underlying graph of the two-terminal graph B'_i which consists of all the edges and real vertices of B_i'. Since $B_i' \subset G_T$ and G is the underlying graph of G_T, the definition of a two-terminal subgraph implies $B_i \subset G$, so that $G \notin \mathbf{Q}$.

Conversely, suppose $G \notin \mathbf{Q}$. Then there exists $B_i \in \mathbf{B}$ such that $B_i \subset G$. Since $G_T = (V, E, x, y)$ is a terminal-attached graph of G, G_T contains all the vertices and edges of B_i. Construct a terminal-attached graph B_i' obtained from B_i by designating the terminals in the following way:

(i) If B_i contains vertex x (or y), then designate it as the first (second) terminal of B_i'.
(ii) Otherwise add to B_i an isolated virtual vertex v_{t1} (v_{t2}) designated as the first (second) terminal.

Observe that $B_i' \subset G_T$ and $B_i' \in \mathbf{B}_T$. Hence $G_T \notin \mathbf{Q}_T$. □

According to Lemma 2, it is sufficient to verify that the property $\mathbf{Q}_T$ on two-terminal graphs satisfies the requirements of Lemma 1. Let a set $\mathbf{S}$ consist of all two-terminal graphs that either belong to $\mathbf{B}_T$ or can be obtained from a two-terminal graph in $\mathbf{B}_T$ by a sequence of series (of type I and type II) and parallel separations. Note that the set $\mathbf{B}_T$ is finite and that any separation graph has not more vertices than the original two-terminal graph. It hence follows that set $\mathbf{S}$ is finite. We will show intuitively that the set of all the properties defined by a subset of $\mathbf{S}$ satisfies the requirements of Lemma 1. In what follows we write $\mathbf{S} = \{S_1, S_2, \ldots, S_\alpha\}$. Furthermore, we define $S[I]$, $\sigma_I(i)$, $\phi_I(i)$, $\sigma_I(J)$, and $\phi_I(J)$ for $i = 1, 2, \ldots, \alpha$ and $I, J \subset \{1, 2, \ldots, \alpha\}$ as follows:

$$S[I] = \{S_j : j \in I\},$$
$$\sigma_I(i) = \{j : S_j \in \mathbf{S} \text{ and } S_i * S_j \in S[I]\},$$
$$\phi_I(i) = \{j : S_j \in \mathbf{S} \text{ and } S_i // S_j \in S[I]\},$$
$$\sigma_I(J) = \bigcup_{j \in J} \sigma_I(j),$$
$$\phi_I(J) = \bigcup_{j \in J} \phi_I(j).$$

We need some more lemmas.

LEMMA 3. *Let G_T be a two-terminal graph such that $G_T = H * K$ (or $H//K$) for some two-terminal graphs H and K, and let G_T' be a two-terminal graph. Then $G_T' \subset G_T$ if and only if there exist two-terminal graphs H' and K' such that $H' \subset H$, $K' \subset K$, and $G_T' = H' * K'$ (or $H'//K'$).*

PROOF. We present the proof only for the necessity of the case in which $G_T = H * K$, since the proof of the remaining case is analogous. Suppose $G_T' \subset G_T$. Let a set V_H' consist of (i) all the vertices of G_T' that belong to H, (ii) virtual terminal v_{t1} (if any) in G_T', and (iii) virtual terminal v_{t2} if either the series connection G_T of H and K is of type II or G_T' does not contain the identified terminal of the series connection G_T. Similarly, we define a set V_K' with respect to K. Since $G_T = H * K$, G_T' is also a series connection of two two-terminal graphs, say, H' and K', having vertex sets V_H' and V_K', respectively. Thus $G_T' = H' * K'$. It should be noted that even if G_T is a series connection of type I, G_T' may be a series connection of type II (if G_T' does not contain the identified terminal of the series connection G_T). Since all the edges and real vertices of H' (or K') belong to $H(K)$, $H' \subset H$ and $K' \subset K$. □

It is sufficient to consider G_T in Lemma 3 as a series connection of type I only to establish our claim of this section, but the result for the case in which G_T is of type II will be used in the succeeding sections. (In the succeeding lemmas, $\mathbf{S}$ is not a fixed set but depends on property $\mathbf{Q}$.)

LEMMA 4. *Let $G_T = H * K$ (or $H//K$) and $S_r \in \mathbf{S}$. Then $S_r \subset G_T$ if and only if there exist $S_f, S_g \in \mathbf{S}$ such that $S_r = S_f * S_g$ (or $S_f//S_g$) and $S_f \subset H$, $S_g \subset K$.*

PROOF. Follows immediately from Lemma 3 and the definition of $\mathbf{S}$. □

Let G_T be a two-terminal graph and $I \subset \{1, 2, \ldots, \alpha\}$. We denote by $S[I] \subset G_T$ the proposition that G_T contains at least one member of set $S[I]$ as a two-terminal subgraph, and denote by $S[I] \not\subset G_T$ the proposition that G_T contains no member of $S[I]$, where the proposition $S[I] \not\subset G_T$ is true if $I = \emptyset$.

LEMMA 5. *Let $G_s = H * K$ (of type I), $G_p = H//K$, and $I \subset \{1, 2, \ldots, \alpha\}$. Then we have*

(i) $S[I] \not\subset G_s$ iff $\bigvee_{J \subset I_s} [(S[J] \not\subset H) \wedge (S[\sigma_I(I_s - J)] \not\subset K)]$, and
(ii) $S[I] \not\subset G_p$ iff $\bigvee_{J \subset I_p} [(S[J] \not\subset H) \wedge (S[\phi_I(I_p - J)] \not\subset K)]$,

where

$$I_s = \{i : \text{there exists } S_j \in \mathbf{S} \text{ such that } S_i * S_j \in S[I]\},$$
$$I_p = \{i : \text{there exists } S_j \in \mathbf{S} \text{ such that } S_i//S_j \in S[I]\}.$$

PROOF. We will establish only (i), since the proof for (ii) is similar. If $I = \emptyset$, our claim follows immediately. Assume $I \neq \emptyset$; then it follows from Lemma 4 and the definitions of $S[I]$, I_s, and σ_I that

$$S[I] \subset G_s \quad \text{iff} \quad \text{there exist } S_f \subset H \text{ and } S_g \subset K \text{ such that } S_f \in S[I_s] \text{ and } S_g \in S[\sigma_I(f)],$$

that is,

$$S[I] \subset G_s \quad \text{iff} \quad \bigvee_{f \in I_s} [(S_f \subset H) \wedge (S[\sigma_I(f)] \subset K)].$$

Taking the negation of both sides of the above, we have

$$S[I] \not\subset G_s \quad \text{iff} \quad \bigwedge_{f \in I_s} [(S_f \not\subset H) \vee (S[\sigma_I(f)] \not\subset K)].$$

Transforming the right-hand side of the above to a disjunctive normal form, we have

$$S[I] \not\subset G_s \quad \text{iff} \quad \bigvee_{J \subset I_s} [(S[J] \not\subset H) \wedge (S[\sigma_I(I_s - J)] \not\subset K)],$$

as desired. □

We are now ready to prove the following theorem.

THEOREM 1. *Let $\mathbf{Q}$ be a property on graphs defined by a finite number of forbidden subgraphs. Then the decision problem with respect to $\mathbf{Q}$ is linear-time computable for every series–parallel graph.*

PROOF. Let G_T be a terminal-attached graph obtained from a series–parallel graph G by designating two ends of any edge of G as the terminal vertices, so that G_T is TTSP. According to Lemma 2, the problem with respect to $\mathbf{Q}$ on a series–parallel graph G can be reduced to the problem with respect to the property $\mathbf{Q}_T$ on the TTSP graph G_T. Therefore it suffices to show that there exists a set of properties associated with $\mathbf{Q}_T$ that satisfies the requirements (i)–(iii) of Lemma 1. Consider the following set $\mathbf{P}$ of properties on two-terminal graphs:

$$\mathbf{P} = \{\text{“}S[I]\text{”} : I \subset \{1, 2, \ldots, \alpha\}\},$$

where "$S[I]$" denotes the property defined by the set $S[I]$ of forbidden two-terminal subgraphs: a two-terminal graph satisfies property "$S[I]$" if it contains no member of $S[I]$ as a two-terminal subgraph. We write $\mathbf{P} = \{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_k\}$, where $k = 2^\alpha$. Note that set $\mathbf{P}$ is finite, since α is a constant. Then we have:

(i) The definition of S implies $\mathbf{B}_\mathrm{T} \subset \mathbf{S}$. Thus property $\mathbf{Q}_\mathrm{T}$ is identical with the property "$S[I]$" with $S[I] = \mathbf{B}_\mathrm{T}$, and hence $\mathbf{Q}_\mathrm{T} \in \mathbf{P}$.
(ii) Since α is finite, one can determine in a constant time whether $S[I] \not\subseteq G_{\min}$ for every $S[I]$.
(iii) It follows from Lemma 5 that for each $\mathbf{Q}_r \in \mathbf{P}$ there exist sets of properties $\{\mathbf{Q}_{h1}, \mathbf{Q}_{h2}, \ldots, \mathbf{Q}_{ht}\} \subset \mathbf{P}$ and $\{\mathbf{Q}_{k1}, \mathbf{Q}_{k2}, \ldots, \mathbf{Q}_{kt}\} \subset \mathbf{P}$ such that

$$G_\mathrm{T} \in \mathbf{Q}_r \quad \text{iff} \quad \bigvee_{i=1}^{t} [(H \in \mathbf{Q}_{hi}) \wedge (K \in \mathbf{Q}_{ki})],$$

for any $G_\mathrm{T} = H * K$ (or $H//K$). □

Even if property Q is defined on graphs by a finite number of forbidden *induced* subgraphs, we have results corresponding to Lemmas 2–5. Thus we have

THEOREM 2. *Let* Q *be a property on graphs defined by a finite number of forbidden induced subgraphs. Then the decision problem with respect to* Q *is linear-time computable for every series–parallel graph.*

4. *Edge Deletion Problems*

In this section we consider the minimum edge deletion problem with respect to a property Q for series–parallel graphs, in which one would like to determine the minimum number of edges of a given series–parallel graph whose deletion results in a graph satisfying Q. We show that this problem is also linear-time computable if Q can be defined by a finite number of forbidden (induced) subgraphs.

For a graph $G = (V, E)$ (or two-terminal graph $G = (V, E, x, y)$) and property Q on (two-terminal) graphs, define $L(G, \mathbf{Q})$ as follows: If there exists a subset E_s of E such that $G - E_s \in \mathbf{Q}$, then $L(G, \mathbf{Q})$ is the minimum cardinality of such a set E_s; otherwise $L(G, \mathbf{Q})$ is undefined (or ∞). The edge deletion problem asks $L(G, \mathbf{Q})$ for a given G and a fixed Q. Let Q, $\mathbf{Q}_\mathrm{T}$, and P be defined as in the preceding section. Then

LEMMA 6. *Let $G = (V, E)$ be a graph, and let $G_T = (V', E', x, y)$ be any terminal-attached graph of G. Then $L(G, \mathbf{Q}) = L(G_T, \mathbf{Q}_T)$.*

PROOF. For a set $E_s \subset E = E'$, define a graph G' by $G' = G - E_s = (V, E - E_s)$, and a two-terminal graph G'_T by $G'_\mathrm{T} = G_\mathrm{T} - E_s = (V', E - E_s, x, y)$. Then G'_T is a terminal-attached graph of G'. Therefore, by Lemma 2 $G' \in \mathbf{Q}$ if and only if $G'_\mathrm{T} \in \mathbf{Q}_\mathrm{T}$. This implies $L(G, \mathbf{Q}) = L(G_\mathrm{T}, \mathbf{Q}_\mathrm{T})$. □

Lemma 6 implies that the minimum edge deletion problem on a graph can be reduced to the same problem on a two-terminal graph.

LEMMA 7. *If G_T is a two-terminal graph such that $G_T = H * K$ (of type I) (or $G_T = H//K$) for two two-terminal graphs H and K, then for each $\mathbf{Q}_r \in \mathbf{P}$,*

$$L(G_T, \mathbf{Q}_r) = \min_{1 \le i \le t} [L(H, \mathbf{Q}_{hi}) + L(K, \mathbf{Q}_{ki})].$$

PROOF. We give a proof only for the case $G_\mathrm{T} = H * K$, since the proof for the case $G_\mathrm{T} = H//K$ is similar. Note that $\mathbf{Q}_\mathrm{T}$ together with P satisfies the hypothesis of Lemma 1. Clearly, the edge set E of G_T is the union of the edge sets of H and K. Let E_s be any subset of E, and let sets E' and E'' consist of all the edges of E_s that belong to H or K, respectively, so that $E_s = E' + E''$. Consider the respective two-terminal subgraphs $G'_T = G_T - E_s$, $H' = H - E'$, and $K' = K - E''$ of G_T, H, and K. By Lemma 3, $G'_T = H' * K'$. By condition (iii) of Lemma 1, $G'_T \in Q_r$ if and only if there exist properties $Q_{hi}, Q_{ki} \in P$, $1 \le i \le t$, such that $H' \in Q_{hi}$ and $K' \in Q_{ki}$. This implies

$$L(G_T, Q_r) = |E_s| = |E'| + |E''| = \min_{1 \le i \le t} [L(H, Q_{hi}) + L(K, Q_{ki})]. \quad \square$$

We now have the following theorem.

THEOREM 3. *Let Q be a property on graphs defined by a finite number of forbidden (induced) subgraphs. Then the minimum edge deletion problem with respect to Q is linear-time computable for every series–parallel graph.*

5. *Vertex Deletion Problems*

In this section we consider the vertex deletion problems with respect to a property Q for series–parallel graphs. For a graph $G = (V, E)$ and property Q on graphs, define $N(G, Q)$ as follows: If there exists $V_s \subset V$ such that $G - V_s \in Q$, then $N(G, Q)$ is the minimum cardinality of such a set V_s; otherwise $N(G, Q)$ is undefined (or ∞). The vertex deletion problem asks $N(G, Q)$ for a given G and a fixed Q.

For a two-terminal graph $G = (V, E, x, y)$, we consider a vertex deletion problem with the additional constraint that one or both of the terminals should be deleted. For a property Q and $m, n = 1$ or 0, define $N(G, Q, m, n)$ as follows: If there exists $V_s \subset V$ such that (i) $G - V_s \in Q$, (ii) $x \in V_s$ iff $m = 1$, and (iii) $y \in V_s$ iff $n = 1$, then $N(G, Q, m, n)$ is the minimum cardinality of such a set V_s; otherwise $N(G, Q, m, n)$ is undefined (or ∞).

Let Q, Q_T, and P be defined as in Section 3. Then we can reduce the problem on an ordinal graph to that on a two-terminal graph as follows.

LEMMA 8. *Let $G = (V, E)$ be a graph, and let $G_T = (V', E', x, y)$ be any terminal-attached graph of G. Then*

$$N(G, Q) = \min_{m,n \in \{0,1\}} N(G_T, Q_T, m, n).$$

PROOF. We now claim that if $V_s \subset V'$ attains the minimum of the right-hand side of the desired equation, then V_s contains no virtual terminals, so $V_s \subset V$. On the contrary, suppose that G_T has a virtual terminal v_{ti} ($i = 1$ or 2) and V_s contains it. Then the two-terminal induced subgraph $G'_T = G_T - V_s$ of G_T also contains v_{ti} by the definition of a two-terminal subgraph. Therefore $G'_T = G_T - (V_s - \{v_{ti}\})$, which contradicts the minimality of V_s.

Let V_s be any subset of V. Consider an induced subgraph $G' = G - V_s$ of G and a two-terminal induced subgraph $G'_T = G_T - V_s$ of G_T. Clearly G'_T is a terminal-attached graph of G'. By Lemma 2, $G' \in Q$ if and only if $G'_T \in Q_T$. This implies the desired equation. $\square$

LEMMA 9

(i) *If G_T is a two-terminal graph such that $G_T = H * K$ (of type I) for two-terminal graphs H and K, then for $Q_r \in P$,*

$$N(G_T, Q_r, m, n) = \min\left[\min_{1 \le i \le t} [N(H, Q_{hi}, m, 0) + N(K, Q_{ki}, 0, n)],\ \min_{1 \le i \le t} [N(H, Q_{hi}, m, 1) + N(K, Q_{ki}, 1, n)] - 1\right].$$

(ii) *If G_T is a two-terminal graph such that $G_T = H//K$ for two-terminal graphs H and K, then*

$$N(G_T, \mathbf{Q}_r, m, n) = \min_{1 \le i \le t} [N(H, \mathbf{Q}_{hi}, m, n) + N(K, \mathbf{Q}_{ki}, m, n)] - (m + n).$$

PROOF. We give a proof only for (i), since the proof for (ii) is similar. Let V_s be a subset of the vertex set of G_T such that $G_T - V_s \in \mathbf{Q}_r$ and $|V_s| = N(G_T, \mathbf{Q}_r, m, n)$, and let sets V' and V'' consist of all the vertices of V_s that belong to H and K, respectively. Then we have $V_s = V' \cup V''$ and $|V' \cap V''| \le 1$ by the definition of a series connection. Now consider the two-terminal subgraph $G'_T = G_T - V_s$ of G_T. We can verify, as in the proof of Lemma 3, that G'_T is a series connection of two two-terminal graphs H' and K' such that $H' \subset H$ and $K' \subset K$ where $H' = H - V'$ and $K' = K - V''$. Thus $G'_T = H' * K'$, where $*$ is of type II if V_s contains the identified terminal of H and K. Note that property $\mathbf{Q}_T$ together with $\mathbf{P}$ satisfies conditions (i)–(iii) of Lemma 1 even if $*$ in condition (iii) means a series connection of type II. Therefore there exist two properties $\mathbf{Q}_{hi}, \mathbf{Q}_{ki} \in \mathbf{P}$, $1 \le i \le t$, such that $H' = H - V' \in \mathbf{Q}_{hi}$ and $K' = K - V'' \in \mathbf{Q}_{ki}$. Thus if $|V' \cap V''| = 0$, then the identified vertex of the series connection G_T is contained neither in V' nor in V'', so we have

$$\begin{aligned} N(G_T, \mathbf{Q}_r, m, n) &= |V_s| = |V'| + |V''| \\ &\ge \min_{1 \le i \le t} [N(H, \mathbf{Q}_{hi}, m, 0) + N(K, \mathbf{Q}_{ki}, 0, n)]. \end{aligned}$$

If $|V' \cap V''| = 1$, then V' contains the second terminal of H and V'' the first terminal of K, both of which are contracted into a single vertex in G_T, so we have

$$\begin{aligned} N(G_T, \mathbf{Q}_r, m, n) &= |V_s| = |V'| + |V''| - 1 \\ &\ge \min_{1 \le i \le t} [N(H, \mathbf{Q}_{hi}, m, 1) + N(K, \mathbf{Q}_{ki}, 1, n)] - 1. \end{aligned}$$

We can similarly establish the converse inequality, so the desired equation follows. □

We now have the following theorem.

THEOREM 4. *Let $\mathbf{Q}$ be a property on graphs defined by a finite number of forbidden (induced) subgraphs. Then the minimum vertex deletion problem with respect to $\mathbf{Q}$ is linear-time computable for every series-parallel graph.*

6. *Properties Characterized by Homeomorphic Subgraphs*

There exist graph properties whose characterizations require an *infinite* number of forbidden subgraphs, but some of them can be characterized by a *finite* number of forbidden homeomorphic subgraphs. For example, planarity is characterized by two forbidden homeomorphic subgraphs K_5 and $K_{3,3}$, while it requires an infinite number of forbidden subgraphs, that is, all the graphs that are homeomorphic to K_5 or $K_{3,3}$, if one characterizes it by forbidden subgraphs instead of homeomorphic subgraphs.

In this section we show that if property $\mathbf{Q}$ is defined by a finite number of forbidden homeomorphic subgraphs, then the decision and minimum edge (vertex) deletion problems are also linear-time computable.

We now present some more terminology. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. G_2 is *homeomorphic* to G_1 if there exist mappings ψ and θ—ψ from E_1 onto a set of paths of G_2 and θ from V_1 into V_2—such that for each edge $(v, w) \in E_1$, path $\psi((v, w))$ has ends $\theta(v)$ and $\theta(w)$, and no two paths $\psi((v_1, w_1))$ and $\psi((v_2, w_2))$ share a vertex except possibly an end of both paths. Next let $G_1 = (V_1, E_1, x_1, y_1)$ and $G_2 = (V_2, E_2, x_2, y_2)$ be any two two-terminal graphs. G_2 is *two-*

terminal homeomorphic to G_1 if (i) the underlying graph of G_2 is homeomorphic to the underlying graph of G_1, (ii) x_2 (or y_2) is virtual if and only if x_1 (or y_1) is virtual, and (iii) the associated mapping θ satisfies $\theta(x_1) = x_2$ and $\theta(y_1) = y_2$. For a two-terminal graph G we denote by Hom(G) the set of all the two-terminal graphs that are two-terminal homeomorphic to G.

Suppose that property **Q** is defined by a finite set of forbidden homeomorphic subgraphs, say $\mathbf{B} = \{B_1, B_2, \ldots, B_q\}$; set **Q** consists of all the graphs that contain no subgraph homeomorphic to any $\mathbf{B}_i$, $i = 1, 2, \ldots, q$. As in Section 5, let $\mathbf{B}_\mathrm{T}$ be the set of all the terminal-attached graphs of $B_1, B_2, \ldots, B_q$. Define a set $\mathbf{B}_h$ of two-terminal graphs as follows:

$$\mathbf{B}_h = \mathbf{B}_\mathrm{T} \cup \mathbf{B}_\mathrm{I} \cup \mathbf{B}_\mathrm{II} \cup \mathbf{B}_\mathrm{III},$$

where

$\mathbf{B}_\mathrm{I}$: the set of all two-terminal graphs obtained from any $B_i \in \mathbf{B}$ by two operations: replace any edge (u, v) of B_i by a new real terminal z together with two edges (u, z) and (z, v); and designate either an arbitrary vertex of B_i different from z or a newly added isolated virtual vertex as the other terminal;

$\mathbf{B}_\mathrm{II}$: the set of all the two-terminal graphs obtained from any $B_i \in \mathbf{B}$ by replacing any two edges (u, v) and (u', v') of B_i by new real terminals x and y together with edges (u, x), (x, v), (u', y), and (y, v'); and

$\mathbf{B}_\mathrm{III}$: the set of all the two-terminal graphs obtained from any $B_i \in \mathbf{B}$ by replacing any edge (u, v) of B_i by new real terminals x and y together with edges (u, x), (x, y), and (y, v).

We now define property $\mathbf{Q}_h$ on two-terminal graphs by the set $\mathbf{B}_h$ of forbidden two-terminal homeomorphic subgraphs: set $\mathbf{Q}_h$ consists of all the two-terminal graphs containing no two-terminal subgraphs that are two-terminal homeomorphic to any member of $\mathbf{B}_h$. Then we have the following lemma.

LEMMA 10. *Let $G_T = (V', E', x, y)$ be a terminal-attached graph of a graph $G = (V, E)$. Then $G \in \mathbf{Q}$ if and only if $G_T \in \mathbf{Q}_h$.*

See the appendix for the proof.

The following result concerning $\mathbf{Q}_h$ corresponds to Lemmas 6 and 8.

LEMMA 11. *If G_T is a terminal-attached graph of a graph $G = (V, E)$, then*

(*i*) $L(G, \mathbf{Q}) = L(G_T, \mathbf{Q}_h)$; *and*

(*ii*) $N(G, \mathbf{Q}) = \min_{m,n\in\{0,1\}} N(G_T, \mathbf{Q}_h, m, n)$.

PROOF. With the aid of Lemma 10 we can establish our claim as in the proof of Lemmas 6 and 8. □

The preceding Lemmas 10 and 11 imply that the decision problem and the minimum edge (vertex) deletion problem, both with respect to property **Q** on a graph, can be reduced to the same problems with respect to property $\mathbf{Q}_h$ on a two-terminal graph. In what follows we verify that $\mathbf{Q}_h$ satisfies the requirement of Lemma 1.

We now define a new series separation in terms of the connection associated with the separation.

Definition 7. Let $H = (V_H, E_H, x, z_2)$ and $K = (V_K, E_K, z_1, y)$ be two-terminal graphs. If both z_1 and z_2 are real vertices and the sum of degrees of z_1 in K and z_2 in

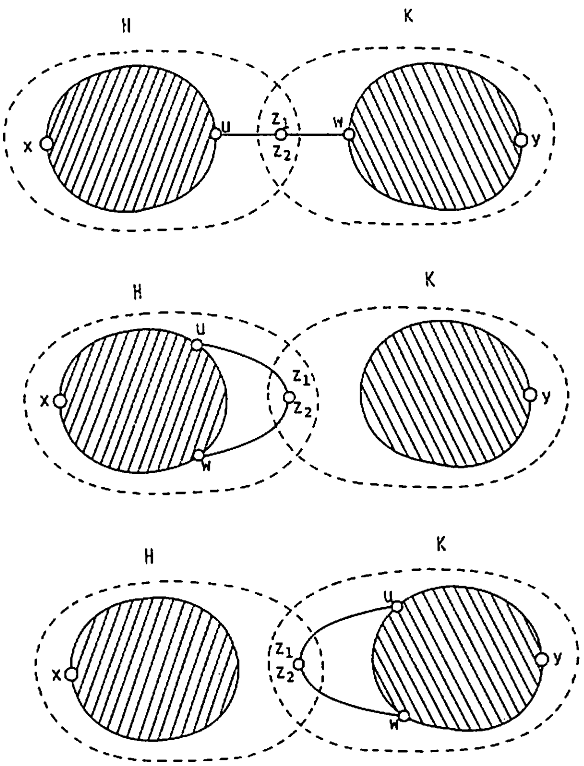


FIG. 5. Series separations of type III.

H is two, then a *series connection of type III* of H and K is a two-terminal graph G_s obtained from the union of H and K by identifying z_1 and z_2 and replacing the two series edges incident to the identified vertex by a single edge. (See Figure 5.) Then H and K are called *series separations of type III* of G_s.

From now on we write $G = H *_h K$ if H and K are series separations of G of type I, II, or III.

We can now define the set S of forbidden two-terminal homeomorphic subgraphs as the set of all the two-terminal graphs that are members of $\mathbf{B}_h$ or can be obtained from a member of $\mathbf{B}_h$ by a sequence of series and parallel separations of all the types. We write $\mathrm{S} = \{S_1, S_2, \ldots, S_\alpha\}$. It should be noted that $\mathbf{B}_h$, and hence S, are finite sets. For $i = 1, 2, \ldots, \alpha$ and a subset I of $\{1, 2, \ldots, \alpha\}$, define $S[I]$ and $\phi_I(i)$ as in Section 3, and define $\sigma_I(i)$ as follows:

$$\sigma_I(i) = \{j : S_j \in \mathrm{S} \text{ and } S_i *_h S_j \in S[I]\}.$$

We write $\mathrm{Hom}(S_r) \subset G_T$ if G_T contains at least one member of the set $\mathrm{Hom}(S_r)$ as a two-terminal subgraph. The following lemma corresponds to Lemma 4 of Section 3.

LEMMA 12. *Let G_T be a two-terminal graph such that $G_T = H * K$ (or $H//K$) where $H = (V_H, E_H, a, b)$ and $K = (V_K, E_K, c, d)$, and let $S_r \in$ S. Then $Hom(S_r) \subset G_T$ if and only if there exist $S_f, S_g \in$ S such that $S_r = S_f *_h S_g$ $(S_f//S_g)$, $Hom(S_f) \subset H$, and $Hom(S_g) \subset K$.*

PROOF. Left to the reader. □

Using the preceding lemma, we obtain the following result, where we write $\mathrm{Hom}(S[I]) \not\subseteq G_s$ if G_s contains no member of set $\mathrm{Hom}(S[I])$ as a two-terminal subgraph.

LEMMA 13. *Let G_s and G_p be two-terminal graphs such that $G_s = H * K$ (of type I) and $G_p = H//K$, and let $I \subset \{1, 2, \ldots, \alpha\}$. Then*

(i) $Hom(S[I]) \not\subseteq G_s$ *iff* $\bigvee_{J \subset I_s} [(Hom(S[J] \not\subseteq H) \wedge (Hom(S[\sigma_I(I_s - J)]) \not\subseteq K)]$; *and*
(ii) $Hom(S[I]) \not\subseteq G_p$ *iff* $\bigvee_{J \subset I_p} [(Hom(S[J]) \not\subseteq H) \wedge (Hom(S[\phi_I(I_p - J)]) \not\subseteq K)]$,

where

$$I_s = \{i: \text{there exists } S_j \in \mathbf{S} \text{ such that } S_i *_h S_j \in S[I]\},$$
$$I_p = \{i: \text{there exists } S_j \in \mathbf{S} \text{ such that } S_i//S_j \in S[I]\}.$$

PROOF. With the aid of Lemma 12 we can establish our claim as in the proof of Lemma 5. □

We now have the following theorem by Lemmas 1 and 10–13.

THEOREM 5. *Let* **Q** *be a property on graphs defined by a finite number of forbidden homeomorphic subgraphs. Then the decision problem and the minimum edge (vertex) deletion problem, both with respect to* **Q**, *are linear-time computable for every series–parallel graph.*

7. *Generalized Matching Problem*

In this section we consider the "generalized matching problem" in which, given an instance graph G and a fixed graph B, one would like to determine the maximum number of (vertex-) disjoint copies of B contained in G. Of course, a special case of this problem is the maximum matching problem, in which B is K_2, the graph of a single edge. Although there exists a polynomial-time algorithm to solve the maximum matching problem [5, 6], the generalized matching problem is NP-complete if B contains a component of at least three vertices [13]. We show that the generalized matching problem is also linear-time computable for every series–parallel graphs whenever B is a connected graph.

Suppose that B is a connected graph. We denote by $M(G)$ the maximum number of disjoint copies of B contained in a graph G. We first show that the generalized matching problem which asks $M(G)$ of a given graph G can be reduced to the same problem on a two-terminal graph. Let $\mathbf{B}_\mathrm{T}$ be the set of all the terminal-attached graphs of B, and let $\mathbf{S}$ be the set of all the two-terminal graphs that are members of $\mathbf{B}_\mathrm{T}$ or obtained from a member of $\mathbf{B}_\mathrm{T}$ by a sequence of the following separations: series separations of types I and II, and parallel separations. We write $\mathbf{S} = \{S_0, S_1, \ldots, S_\alpha\}$, where S_0 denotes a two-terminal graph consisting of only two isolated virtual terminals with no edges, and S_1 denotes a terminal-attached graph of B having two isolated virtual terminals (See Figure 6). Since B is connected, S_i contains at least one real terminal unless $i = 0$ or 1. We define $S[I]$, $\sigma_I(i)$, and $\phi_I(i)$ as in Section 3 and simply write $\sigma_j(i)$ and $\phi_j(i)$ instead of $\sigma_{\{j\}}(i)$ and $\phi_{\{j\}}(i)$, respectively. A two-terminal graph $G_1 = (V_1, E_1, x_1, y_1)$ is *two-terminal isomorphic* to a two-terminal graph $G_2 = (V_2, E_2, x_2, y_2)$ if there exists a one-to-one mapping m from V_1 onto V_2 such that (i) $m(x_1) = x_2$, (ii) $m(y_1) = y_2$, and (iii) $(v, w) \in E_1$ iff $(m(v), m(w)) \in E_2$. Two two-terminal graphs are said to be *disjoint* if they have no common *real* vertices. If G_T is a two-terminal graph and S_f and S_g are two-terminal graphs such that $S_f, S_g \in \mathbf{S}$ and $S_f, S_g \neq S_1$, then define $M(G_\mathrm{T}; S_f, S_g)$ as follows: If G_T contains two disjoint two-terminal subgraphs of which one is two-terminal isomorphic to S_f and the other to S_g, then $M(G_\mathrm{T}; S_f, S_g)$ is the maximum cardinality of a set of disjoint two-terminal subgraphs of G_T such that two of them are S_f and S_g above and all the others are copies of S_1; otherwise $M(G_\mathrm{T}; S_f, S_g)$ is undefined (or $-\infty$). That is,

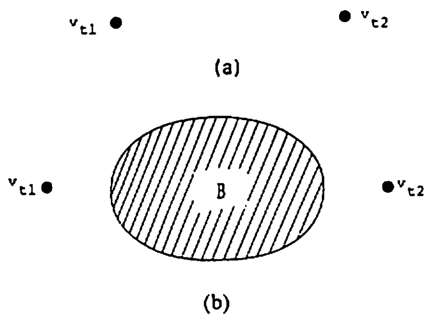


FIG. 6. Two-terminal graphs (a) S_0 and (b) S_1.

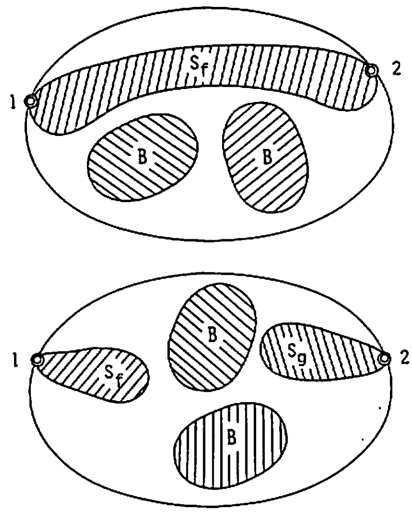


FIG. 7. Possible arrangements of two-terminal subgraphs S_f, S_g, and S_1's in a two-terminal graph with terminals 1 and 2.

$M(G_T;\, S_f,\, S_g)$ denotes the maximum number of disjoint subgraphs contained in the underlying graph of G_T, two of which are S_f and S_g, both containing a terminal of G_T, and all the others of which are B's containing no terminals of G_T (See Figure 7). One or both of the parameters S_f and S_g are possibly empty in $M(G_T;\, S_f,\, S_g)$. We write simply $M(G_T)$ if the parameters S_f and S_g are both empty, or $M(G_T;\, S_f)$ if the parameter S_g is empty. The definition above implies that $M(G_T;\, S_f,\, S_g)$ is necessarily undefined if neither S_f nor S_g contains a virtual terminal, while $M(G_T;\, S_f)$ is not necessarily undefined even if S_f contains no virtual terminal.

We now show that the generalized matching problem on a two-terminal graph G_T is included in an extended generalized matching problem which asks all $M(G_T;\, S_f,\, S_g)$'s. The definition of $\mathbf{S}$ implies $\mathbf{B}_T \subset \mathbf{S}$. Let $\mathbf{B}'_T = \mathbf{B}_T - \{S_1\}$; then we have the following lemma.

LEMMA 14. *Let $G = (V, E)$ be a graph, and let $G_T = (V', E', x, y)$ be any terminal-attached graph of G. Then we have*

$$M(G) = max\left[M(G_T),\ \max_{S_f \in \mathbf{B}'_T} M(G_T;\, S_f),\ \max_{S_f, S_g \in \mathbf{B}'_T} M(G_T;\, S_f,\, S_g) \right].$$

PROOF. Suppose that $\{B_1, B_2, \ldots, B_t\}$ is a collection of t disjoint subgraphs of G, each of which is isomorphic to B. With each B_i, $i = 1, 2, \ldots, t$, one can associate a two-terminal subgraph B'_i of G_T as follows: If B_i contains a vertex of G that is a terminal of G_T, then designate it as a terminal of B'_i; otherwise, add an isolated virtual terminal to B_i. Then clearly the B_i' ($i = 1, 2, \ldots, t$) are disjoint two-terminal subgraphs of G_T. Note that each B_i', $i = 1, 2, \ldots, t$, is two-terminal isomorphic to S_1 or a member of $\mathbf{B}_T'$, and that every member of $\mathbf{B}_T'$ contains at least one real terminal. Hence at most two of $B_1, B_2, \ldots, B_t$ are two-terminal isomorphic to a member of $\mathbf{B}_T'$. There are three cases: either none of them, exactly one, or exactly two are isomorphic to a member of $\mathbf{B}_T'$. Each case corresponds to one of the three terms of the right-hand side of the desired equation. Thus we have the desired equation. □

The preceding lemma implies that the generalized matching problem on a graph G can be reduced to an extended generalized matching problem on a two-terminal graph G_T which asks $M(G_T; S_f, S_g)$ for all $S_f, S_g \in \mathbf{B}_T'$, $S_f, S_g \neq S_1$. We now show that if G_T is a series or parallel connection of two two-terminal graphs H and K, then the solution to G_T can be efficiently obtained from the solutions to H and K.

LEMMA 15. *Let G_T be a two-terminal graph such that $G_T = H * K$ (of type I), and let $S_f, S_g \in \mathbf{S}$ and $S_f, S_g \neq S_1$. Then we have*

(*i*)

$$M(G_T) = max\left[M(H) + M(K),\ \max_{i \in I_1'}\left[M(H; S_i) + \max_{i' \in \sigma_1(i)} M(K; S_{i'}) - 1\right]\right];$$

(*ii*) *$M(G_T; S_f)$ is undefined ($-\infty$) if $I_f = \varnothing$ (i.e., S_f is not series separable); otherwise,*

$$M(G_T; S_f) = max\left[\max_{i \in I_f}\left[M(H; S_i) + \max_{i' \in \sigma_f(i)} M(K; S_{i'}) - 1\right],\ \max_{\substack{i \in I_1' \\ j \in I_f}}\left[M(H; S_i, S_j) + \max_{\substack{i' \in \sigma_1(i) \\ j' \in \sigma_f(j)}} M(K; S_{i'}, S_{j'}) - 2\right]\right];$$

(*iii*) *$M(G_T; S_f, S_g)$ is undefined ($-\infty$) if $I_f = \varnothing$ or $I_g = \varnothing$; otherwise,*

$$M(G_T; S_f, S_g) = max\left[\max_{i \in I_1'}\left[M(H; S_f, S_i) + \max_{i' \in \sigma_1(i)} M(K; S_g, S_{i'}) - 1\right],\ \max_{i \in I_1'}\left[M(H; S_g, S_i) + \max_{i' \in \sigma_1(i)} M(K; S_f, S_{i'}) - 1\right],\ \max_{\substack{i \in I_f \\ j \in I_g}}\left[M(H; S_i, S_j) + \max_{\substack{i' \in \sigma_f(i) \\ j' \in \sigma_g(j)}} M(K; S_{i'}, S_{j'}) - 2\right]\right],$$

where

$I_1' = \{i: \text{there exists } S_j \in \mathbf{S} \text{ such that } S_i \neq S_0, S_1 \text{ and } S_i * S_j = S_1\}$,
$I_f = \{i: \text{there exists } S_j \in \mathbf{S} \text{ such that } S_i * S_j = S_f\}$,
$I_g = \{i: \text{there exists } S_j \in \mathbf{S} \text{ such that } S_i * S_j = S_g\}$.

PROOF. Left to the reader. □

LEMMA 16. *Let G_T be a two-terminal graph such that $G_T = H//K$, and let $S_f, S_g \in \mathbf{S}$ and $S_f, S_g \neq S_1$. Then we have*

(i) $M(G_T) = M(H) + M(K)$;
(ii) $M(G_T; S_f) = max_{i \in I_f} [M(H; S_i) + max_{i' \in \phi_f(i)} M(K; S_{i'}) - 1]$;
(iii) $M(G_T; S_f, S_g) = max_{i \in I_f, j \in I_g} [M(H; S_i, S_j) + max_{i' \in \phi_f(i), j' \in \phi_g(j)} M(K; S_{i'}, S_{j'}) - 2]$,

where

$$I_f = \{i: \text{there exists } S_j \in \mathrm{S} \text{ such that } S_i // S_j = S_f\},$$
$$I_g = \{i: \text{there exists } S_j \in \mathrm{S} \text{ such that } S_i // S_j = S_g\}.$$

PROOF. Since B is connected, either $S_1 \subset H$ or $S_1 \subset K$ if $S_1 \subset G_p$. Noting this fact, one can verify our claim. □

It should be noted that none of S_i, S_j, $S_{i'}$, and $S_{j'}$ is identical with S_1 in the right-hand side of the equations of Lemmas 15 and 16.

We now have the following theorem by using the preceding two lemmas.

THEOREM 6. *The generalized matching problem with respect to a connected graph B is linear-time computable for every series–parallel graph.*

8. *Conclusion*

In this paper we showed, in a unified manner, that if an input graph is restricted to the class of series–parallel graphs, then there exist linear-time algorithms for many combinatorial problems, including (i) decision problems, (ii) minimum edge (vertex) deletion problems both with respect to properties characterized by a finite number of forbidden (induced or homeomorphic) subgraphs, and (iii) generalized matching problems. Consequently, the following problems, among others, proved to be linear-time computable for the class of series–parallel graphs:

(1) the minimum vertex cover problem (equivalently, the maximum independent vertex set problem);
(2) the maximum (induced) line-subgraph problem;
(3) the minimum edge (vertex) deletion problem with respect to property "without cycles (or paths) of specified length n or any length $\leq n$";
(4) the maximum outerplanar (induced) subgraph problem;
(5) the minimum feedback vertex set problem;
(6) the maximum ladder (induced) subgraph problem ($K_{2,3}$ and its dual are the forbidden homeomorphic subgraphs of a ladder graph [20]);
(7) the minimum path cover problem (in which one would like to find a minimum number of disjoint paths which contain all the vertices of a given graph);
(8) the maximum matching problem; and
(9) the maximum disjoint triangle problem.

Problems (1)–(3) can be formulated as a minimum edge (vertex) deletion problem with respect to a property defined by a finite number of forbidden (induced) subgraphs, and problems (4)–(7) by forbidden homeomorphic subgraphs, while problems (8) and (9) are examples of the generalized matching problem.

Although there remain a number of combinatorial problems which can not be treated systematically, it is easy to verify individually that the following problems are linear-time computable for series–parallel graphs via an algorithm similar to one in Lemma 1:

(i) the maximum cycle problem (consider two kinds of subgraphs in each decomposed subgraph: the maximum cycle and the longest path connecting the terminals);

(ii) the maximum Eulerlian (induced) subgraph problem (equivalently, the Chinese Postman Problem);
(iii) the maximum bipartite (induced) subgraph problem (equivalently, the maximum cut problem);
(iv) the dominating set problem [12];
(v) the minimum spanning tree problem; and
(vi) the Steiner tree problem.

Furthermore, our results may be generalized in the following directions:

(a) a maximum weight (induced) subgraph problem on series–parallel graphs with weights on the edges or vertices;
(b) directed series–parallel graphs; and
(c) extended series–parallel graphs, for example, n-terminal series-parallel graphs [18].

Finally, we add some remarks on our general techniques. All our algorithms run in time linear on the size of an input graph but take time exponential on the size of the collection of forbidden graphs (or the size of the fixed graph of the generalized matching problem). This fact means that the direct application of our general technique does not always yield an algorithm more practical than an ad hoc one for a particular problem.

Appendix. Proof of Lemma 10

Suppose $G_T \notin \mathbf{Q}_h$. Then G_{T} contains a two-terminal subgraph G'_{T} which is two-terminal homeomorphic to some $B'_i \in \mathbf{B}_h$ under a mapping, say θ. Let G' be the underlying graph of G'_{T}. Define a graph B_i as follows: If $B'_i \in \mathbf{B}_{\mathrm{T}}$, then B_i is the underlying graph of B'_i; and if $B'_i \in \mathbf{B}_{\mathrm{I}} \cup \mathbf{B}_{\mathrm{II}} \cup \mathbf{B}_{\mathrm{III}}$, then B_i is the graph in $\mathbf{B}$ from which B'_i is formed. Then clearly $B_i \in \mathbf{B}$. One can easily show that G' is homeomorphic to B_i under the mapping θ or a restriction of θ. Clearly $G' \subset G$, since $G'_{\mathrm{T}} \subset G_{\mathrm{T}}$. Hence G contains a subgraph homeomorphic to $B_i \in \mathbf{B}$, so that $G \notin \mathbf{Q}$.

For the converse, suppose that $G \notin \mathbf{Q}$. Then G contains a subgraph G' which is homeomorphic to some $B_i \in \mathbf{B}$ under a mapping θ. Construct a terminal-attached graph G'_{T} of G' by designating its terminals as follows: If the terminal-attached graph G_{T} of G contains a real terminal z ($z = x, y$) which is also contained in G', then designate z as a real terminal of G'_{T}; otherwise, add a new isolated virtual terminal z to G'. Then, clearly, $G'_{\mathrm{T}} \subset G_{\mathrm{T}}$. Now construct a two-terminal graph B'_i from B_i as follows. Suppose first that a terminal z of G'_{T} is virtual; then add an isolated virtual terminal z to B_i. Suppose next that a terminal z of G'_{T} is a real vertex. Then the vertex z also exists in the underlying graph G' of G'_{T}. Since G' is homeomorphic to B_i, in G', z is either (i) an end of a path mapped from an edge, say (u, v), of B_i, or (ii) an internal vertex of the path. In case (i), designate the vertex $\theta^{-1}(z)$ as a terminal of B'_i. In case (ii), replace (u, v) of B_i by a new terminal vertex z together with edges (u, z) and (z, v) (or by two new terminal vertices x and y together with edges (u, x), (x, y), and (y, v) if both terminal vertices of G'_{T} are internal vertices of a single path mapped from edge (u, v) of B_i). Obviously G'_{T} is two-terminal homeomorphic to the resulting two-terminal graph B'_i under the mapping θ or an extension of θ. Furthermore, the definition of $\mathbf{B}_h$ implies that $B'_i \in \mathbf{B}_h$. Hence we have $G_{\mathrm{T}} \notin \mathbf{Q}_h$. □

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REFERENCES

1. AHO, A.V., HOPCROFT, J.E., AND ULLMAN, J.D. *The Design and Analysis of Computer Algorithms.* Addison-Wesley, Reading, Mass., 1974.
2. BOULALA, M., AND UHRY, J. Polytope des independants d'un graphe series-parallel. *Discrete Math. 27* (1979), 225–243.
3. CHARTRAND, G., GELLER, D., AND HEDETNIEMI, S. Graphs with forbidden subgraphs. *J. Combinatorial Theory 10* (1971), 12–41.
4. DUFFIN, R.J. Topology of series-parallel networks. *J. Math. Applic. 10* (1965), 303–318.
5. EDMONDS, J. Paths, trees, and flowers. *Canad. J. Math. 17* (1965), 449–467.
6. EVEN, S., AND KARIV, O. An $O(n^{2.5})$ algorithm for maximum matching in general graphs. Proc. 16th IEEE Symp. on Foundations of Computer Science, Berkeley, Calif., 1975, pp. 100–112.
7. GAREY, M.R., JOHNSON, D.S., AND STOCKMEYER, L. Some simplified NP-complete graph problems. *Theor. Comput. Sci.* 1 (1976), 237–267.
8. HADLOCK, F.O. Finding a maximum cut of a planar graph in polynomial time. *SIAM J. Comput. 4* (1975), 221–225.
9. HARARY, F. *Graph Theory.* Addison-Wesley, Reading, Mass., 1969.
10. HOPCROFT, J.E., AND TARJAN, R.E. Dividing a graph into triconnected components. *SIAM J. Comput. 2,* 3 (1973), 135–158.
11. HEDETNIEMI, S.T. Hereditary properties of graphs. *J. Combinatorial Theory B 14* (1973), 94–99.
12. KIKUNO, T., YOSHIDA, N., AND KAKUDA, Y. Dominating set in planar graphs, Tech. Rep. AL79-9, Institute of Electronics and Communication Engineers of Japan, Tokyo, Japan, 1979, pp. 21–30.
13. KIRKPATRICK, D.G., AND HELL, P. On the completeness of a generalized matching problem. Proc. 10th ACM Symp. on Theory of Computing, San Diego, Calif., 1978, pp. 240–245.
14. KRISHNAMOORTHY, M.S., AND DEO, N. Node-deletion NP-complete problems. *SIAM J. Comput. 8,* 4 (1979), 619–625.
15. LEWIS, J.M. On the complexity of the maximum subgraph problem. Proc. 10th ACM Symp. on Theory of Computing, San Diego, Calif., 1978, pp. 265–274.
16. MONMA, C.L., AND SIDNEY, J.B. Sequencing with series-parallel procedure constraints. Tech. Rep. 347, School of Operations Research and Industrial Engineering, Cornell Univ., Ithaca, N.Y., 1979.
17. MOORE, E.F., AND SHANNON, C.E. Reliable circuits using reliable relays, I–II. *J. Franklin Inst. 262,* 3 (1956), 191, and 4 (1956), 281.
18. NISHIZEKI, T., AND SAITO, N. Necessary and sufficient condition for a graph to be three-terminal series-parallel-cascade. *J. Combinatorial Theory B 24,* 3 (1978), 344–361.
19. NISHIZEKI, T., TAKAMIZAWA, K., AND SAITO, N. Algorithms for detecting series-parallel graphs and D-charts. *Trans. Inst. Elect. Commun. Eng. Japan 59,* 3 (1976), 259–260.
20. TOMIZAWA, N. On a specialization sequence from general matroids to ladder graphs with special emphasis on the characterization of ladder matroids. *RAAG Research Notes, Third Series, 191,* Research Assoc. of Applied Geometry, Tokyo, Japan, 1973.
21. VALDES, J., TARJAN, R.E., AND LAWLER, E.L. The recognition of series-parallel digraph. Proc. 11th Ann. ACM Symp. on Theory of Computing, Atlanta, Ga., 1979, pp. 1–12.
22. WATANABE, T., AE, T., AND NAKAMURA, A. On the node cover problem of planar graphs. Proc. of 1979 Int. Symp. on Circuits and Systems, Tokyo, Japan, 1979, pp. 78–81.
23. YANNAKAKIS, M. Node- and edge-deletion NP-complete problems. Proc 10th ACM Symp. on Theory of Computing, San Diego, Calif., 1978, pp. 253–264.

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