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Linear Algorithms for Convex Drawings of Planar Graphs

Norishige Chiba
Tadashi Yamanouchi
Takao Nishizeki

ABSTRACT

In a convex drawing of a planar graph, all the edges are drawn by straight line segments in such a way that every face boundary is a convex polygon. In this paper, we present two linear algorithms for the convex drawing problem of planar graphs: drawing and testing algorithms. The former draws a given planar graph G convex if possible: it extends a given convex polygon of an outer facial cycle of G into a convex drawing of G. The latter tests the possibility: it determines whether G has an outer facial cycle extendable to a convex drawing of G. Moreover it finds all these facial cycles. These two algorithms together can efficiently yield various convex drawings of a planar graph.

1. Introduction

The problem of drawing a planar graph often arises in Design Automation for electrical networks, particularly VLSI circuits. However, we do not here concentrate on a particular practical application. We are interested in drawing a planar graph so that one can easily and rapidly recognize its structure such as the adjacency of vertices and the connectivity-property. One of the feasible methods for the purpose is a convex drawing in which all the edges are drawn by straight line segments without any crossing so that all the face boundaries are convex polygons.

Clearly not every planar graph has a convex drawing. Tutte [5] proved that every 3-connected planar graph has a convex drawing, and established a necessary and sufficient condition for a planar graph to have a convex drawing with prescribed outer polygon. Furthermore he gave a "barycentric mapping" method for finding a convex drawing, which solves a system of linear equations and so usually requires $O(n^3)$ computation time [6]. Throughout this paper we denote by n the number of vertices in a graph.

In this paper we present two linear algorithms for the convex drawing problem of planar graphs: drawing and testing algorithms. The former draws a given planar graph G convex if possible: it extends a given convex polygonal drawing of an outer facial cycle of G into a convex drawing of G. The latter tests the possibility. That is, it determines whether a given planar graph has a convex drawing or not, and moreover finds all the outer facial cycles extendable to a convex drawing of the graph. These two algorithms together yield various convex drawings of a given planar graphs. Both the time and space of these algorithms are linear, so optimal to within a constant factor. Our linear drawing algorithm compares favorable with the known $O(n^3)$ one. The algorithms have been implemented in PASCAL. In Section 5 we give a computational example: a convex drawing of an input planar graph and all its extendable facial cycles.

Recently Thomassen gave a short proof to a strong version of the Tutte's Result [4]. Our drawing algorithm is based on it. On the other hand we modify their results into a form suitable for the convex testing, which is represented in terms of 3-connected components. Using the form, we show that the convex testing of a graph G can be reduced to the planarity testing of a certain graph obtained from G. Our testing algorithm employs this fact together with two linear algorithms of Hopcroft and Tarjan [2] [3]: one for testing the planarity of a graph; and the other for dividing a graph into 3-connected components.

2. Preliminaries

In this section, we first define some terms, then give illustrative examples, and finally present a known result on the convex drawing.

Let $G=(V,E)$ be a *graph* with vertex set V and edge set E. The vertex set of a graph G is often denoted by $V(G)$. We consider only a drawing of a *2-connected simple graph*, that is, a graph with no cut vertices, multiple edges or loops. A graph is *planar* if it is embeddable in the plane without edge crossing. A *plane* graph G is a planar graph which is embedded in the plane. A plane graph divides the plane into connected regions called *faces*. The unbounded face is called the *outer face* of G. A cycle C of a planar graph G is *(outer) facial* if C bounds an (outer) face of a plane graph G. A path joining vertices x and y is called an *x-y path*. A *convex drawing* of a planar graph is a representation of the graph on the plane such that all edges are drawn by straight line segments without any crossing and that all the face boundaries are convex polygons. Since all the edges are drawn by straight lines, a convex drawing of a plane graph is uniquely determined only by the positions of the vertices.

Clearly not every 2-connected planar graph has a convex drawing. For example, the 2-connected planar graph depicted in Figure 1(a) cannot be drawn convex even if any facial cycle is chosen as an outer cycle. Next consider the graph G in Figure 1(b). In this case it depends on the facial cycle chosen as an outer cycle whether G can be drawn convex or not. If the outer facial cycle S is 1-2-3-4-1, G cannot be drawn convex for any polygonal drawing S^* of S, as shown in Figure 1(c). On the other hand, G can be drawn convex if the facial cycle $S=1-2-3-4-5-6-1$ is chosen as an outer cycle and moreover S is drawn as a convex polygon S^* such that vertices 1, 2, 3, 4, and 6 are the apices (i.e. geometric vertices) of S^*, as shown in Figure 1(d). However, if S^* is a convex polygon with the apices 1, 2, 3, and 4, then G cannot be drawn convex as shown in Figure 1(e). Thus we may define: a convex polygonal drawing (for short, a convex polygon) S^* of a facial cycle S of a graph G is *extendable* if there exists a convex drawing of G having S^* as the outer polygon; a facial cycle S is *extendable* if S has an extendable convex polygon S^*.

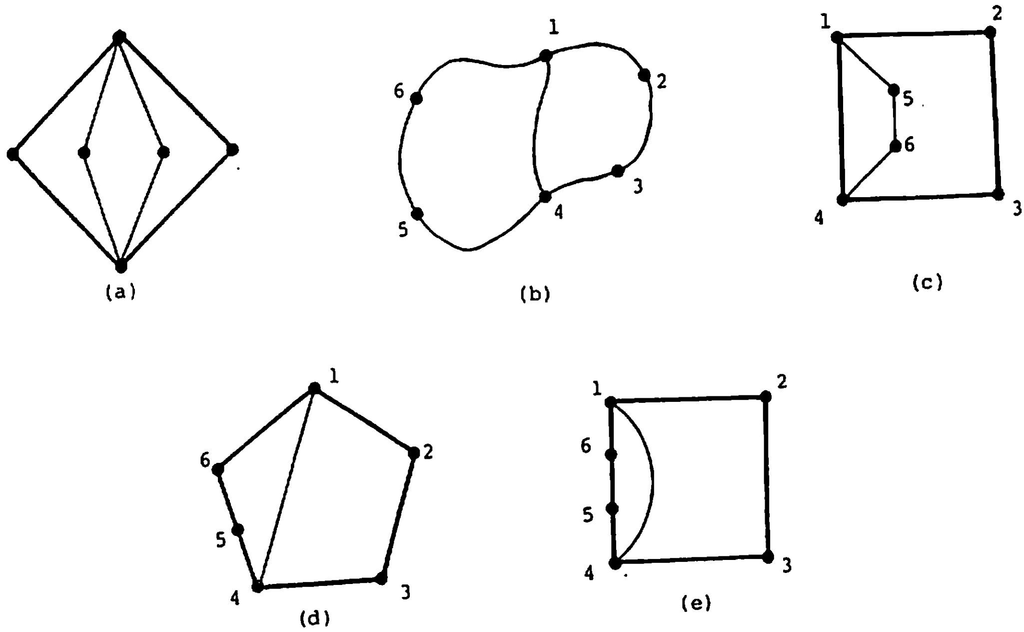


Figure 1. Examples of drawings.

Tutte established a necessary and sufficient condition for a convex polygon to be extendable [5]. The following lemma is a strong version of his result obtained by Thomassen [4].

LEMMA 1. (Thomassen [4]) Let G be a 2-connected plane graph with the outer facial cycle S, and let S^* be a convex polygon of S. Let $P_1, P_2, \ldots, P_k$ be the paths in S, each corresponding to a side of S^*. (Thus S^* is a k-gon. It should be noted that not every vertex of the cycle S is an apex of the polygon S^*.) Then S^* is extendable if and only if Condition I below holds.

CONDITION I

(a) for each vertex v of $G-V(S)$ having degree at least three in G, there exist three paths disjoint except v, each joining v and a vertex of S;

(b) $G-V(S)$ has no connected component C such that all the vertices on S adjacent to vertices in C lie on a single path P_i; and no two vertices in each P_i are joined by an edge not in S; and

(c) any cycle of G which has no edge in common with S has at least three vertices of degree ≥ 3 in G.

3. Convex Drawing Algorithm

In this section we give a linear convex drawing algorithm of planar graphs. Suppose that a 2-connected plane graph G is given together with an extendable convex polygon S^* of the outer facial cycle S. The algorithm extends S^* into a convex drawing of G in linear time.

Our drawing algorithm is based on Thomassen's short proof of Lemma 1. The outline of the drawing algorithm is as follows. We reduce the convex drawing of G to those of several subgraphs of G as follows: delete from G an arbitrary apex v of S^* together with the edges incident to v; divide the resulting graph $G'=G-v$ into the blocks $B_1, B_2, \ldots, B_p$, $p \geq 1$ (see Figure 2); determine a convex polygon S_i^* of the outer facial cycle S_i of each B_i so that B_i with S_i^* satisfies Condition I; and recursively apply the algorithm to each B_i with S_i^* to determine the positions of vertices not in S_i. The detail of our algorithm is as follows.

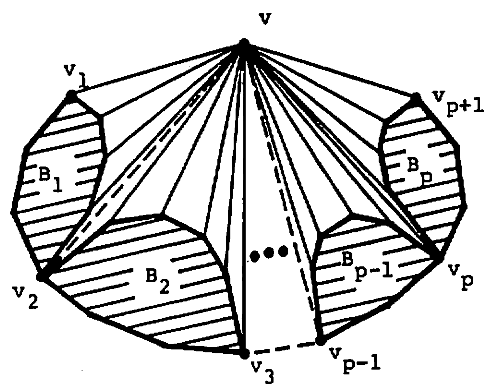


Figure 2. Reduction of the convex drawing of G into subproblems.

Algorithm Convex-Drawing. Let G be a given 2-connected plane graph with the outer facial cycle S, and let S^* be an extendable convex polygon of S. For simplicity we reduce the drawing of the graph G into that of a graph G' which has no vertex of degree two not on S.

STEP 1. For each vertex v of degree two not on S, replace v together with the two edges incident to v by a single edge joining the vertices adjacent to v. Let G' be the resulting graph.

STEP 2. Call the procedure DRAW (G',S,S^*) below to extend S^* into a convex drawing of G'.

STEP 3. For each deleted vertex of degree 2 determine its position on the straight line segment joining the two vertices adjacent to the vertex. *Stop*.

DRAW (G,S,S^*). *Comment:* This procedure extends a convex polygon S^* of the outer facial cycle S of a plane graph G into a convex drawing of G, where G has no vertex of degree 2 not on S.

STEP 1. If G has at most three vertices, a convex drawing of G has been obtained, so *return*. Thus assume that G has at least four vertices. Select an arbitrary apex v of S^*, and let $G':=G-v$. Divide the plane graph G' into the blocks B_i $(1 \le i \le p)$. Let v_1 and v_{p+1} be the two vertices on S adjacent to v, and let v_i, $2 \le i \le p$, be the cut vertices of G' such that $v_1 \in V(B_1)$, $v_{p+1} \in V(B_p)$ and $v_i = V(B_{i-1}) \cap V(B_i)$. (See Figure 2.) Every v_i, $1 \le i \le p+1$, is necessarily on S since the extendable S^* with G satisfies Condition I and G has no vertex of degree two not on S.

STEP 2. Draw each block B_i convex by applying the following procedures.

STEP 2.1. Determine a convex polygon S_i^* of the outer facial cycle S_i of B_i as follows. Since the positions of the vertices in $V(S_i) \cap V(S)$ have been already determined on S^*, one should determine the positions of the vertices in $V(S_i)-V(S)$. Locate the vertices in $V(S_i)-V(S)$ in the interior of the triangle $v \cdot v_i \cdot v_{i+1}$ in such a way that the vertices adjacent to v are apices of a convex polygon S_i^* and the others are on the straight line segments of S_i^*.

STEP 2.2. Recursively call the procedure DRAW(B_i,S_i,S_i^*) to extend S_i^* to a convex drawing of B_i. (Note that the convex polygon S_i^* with B_i satisfies Condition I and B_i has no vertex of degree two not on S_i.)

STEP 3. *Return.*

We have the following result on the algorithm.

THEOREM 1. Let G be a 2-connected plane graph with the outer facial cycle S, and let S^* be an extendable convex polygon of S. Then the algorithm CONVEX-DRAWING extends S^* into a convex drawing of G, and uses linear time and space.

PROOF. Clearly the boundaries of all the inner faces containing the apex v are convex polygons. Therefore, in order to prove inductively the correctness of the algorithm, one should show that every block B_i with S_i^* satisfies Condition I. We omit the proof since it is similar to that of Theorem 5.1 in [4]. Thus we shall establish the claims on time and space.

As a data structure to represent a plane graph $G=(V,E)$, we use doubly linked adjacency lists, in each of which the edges adjacent to a vertex are stored in the order of the plane embedding, clockwisely around the vertex. The two copies of each edge (v,w) in the adjacency lists of v and w are linked each other so that one can be accessed directly from the other. Given an edge e, one can directly access to the edge clockwisely next to e around an end vertex of e. Clearly this data structure requires linear space.

Clearly Steps 1 and 3 of CONVEX-DRAWING can be executed in $O(n)$ time. Therefore we shall prove that Step 2, and so DRAW, spends linear time. Let P be the v_1-v_{p+1} path in the outer facial cycle S' of $G-v$ which newly appears on S'. While traversing P, one can easily (1) find the cut vertices v_i, $2 \le i \le p$, which are also on S, (2) obtain the outer facial cycle S_i of B_i as the union of the traversed

$v_i - v_{i+1}$ path and the $v_{i+1} - v_i$ path on S, and (3) decides the positions of the vertices of S_i as specified in Step 2.1. Thus, we can implement the algorithm so that the time required by the procedure DRAW, exclusive of recursive calls to itself, is proportional to the number of the traversed edges in P, that is, the edges newly appeared on the boundaries of the outer facial cycles. Since every edge appears on a boundary of an outer facial cycle at most once, the number of edges traversed during an execution of DRAW is at most $|E|$ in total. Thus DRAW runs in linear time.

REMARK: In Theorem 1 above we assume that an arithmetic operation of infinite decimal requires one unit time. Tutte's proof of Theorem II in [5] yields another convex drawing algorithm of polynomial time complexity, but possibly of higher order.

4. Testing Algorithm

In this section, we present a convex testing algorithm which determines whether a given 2-connected planar graph has a convex drawing and moreover finds all the extendable facial cycles, both in linear time.

One can construct a linear algorithm which only determines whether a given convex polygon of a particular outer facial cycle is extendable, that is, satisfies Tutte's condition [5] or Condition I. However a planar graph G may have an exponential number of facial cycles so it is impractical to test all the facial cycles of a graph one by one through the algorithm. Note that the plane embedding of G is not always unique unless G is 3-connected [1]. Thus we shall modify Condition I in Lemma 1 into a form suitable for our purpose. One may easily notice that the existence of a convex drawing of a graph G heavily depends on the structure of 3-connected components of G.

This section is organized as follows: Section 4.1 gives definitions of 3-connected components and separation pairs. In Section 4.2 we represent Condition I in terms of 3-connected components. Section 4.3 gives a linear convex testing algorithm. In Section 4.4 we show how to find all the extendable facial cycles of a graph.

4.1. Definitions

We first borrow the definition of some terms from [2]. In Section 4.1 "graph" is used instead of "multigraph" since only multigraphs are concerned. A pair $\{x,y\}$ of vertices of a 2-connected graph $G=(V,E)$ is a *separation pair* if there exist two subgraphs $G'_1=(V_1,E'_1)$ and $G'_2=(V_2,E'_2)$ satisfying the following conditions (a) and (b):

(a) $V=V_1 \cup V_2$, $V_1 \cap V_2 = \{x,y\}$;

(b) $E=E'_1 \cup E'_2$, $E'_1 \cap E'_2=\emptyset$, $|E'_1| \geq 2$, $|E'_2| \geq 2$.

A 2-connected graph G is said to be *3-connected* if G has no separation pair. For a separation pair $\{x,y\}$, $G_1=(V_1,E'_1+(x,y))$ and $G_2=(V_2,E'_2+(x,y))$ are called *split graphs* of G. The new edges (x,y) added to G_1 and G_2 are called *virtual edges*. Dividing a graph G into two split graphs G_1 and G_2 is called *splitting*. Reassembling the two split graphs G_1 and G_2 into G is called *merging*. Merging is the inverse of splitting. Suppose a graph G is split, the split graphs are split, and so on, until no more splits are possible (each remaining graph is 3-connected). The graphs constructed in this way are called the *split components* of G. The split components of a graph G are of three types: *triple bonds* (i.e. a set of three multiple edges), *triangles* (i.e. a cycle consisting of three edges), and 3-connected graphs. The *3-connected components* of G are obtained from the split components of G by merging triple bonds into a bond and triangles into a ring, as far as possible. Here a *bond* is a set of multiple edges, and a *ring* is a cycle (we use "ring" instead of "polygon" used in [2] in order to avoid the confusion). The split components of a graph G are not necessarily unique, but the 3-connected components of G are unique.

We illustrate the decompositions of a 2-connected graph in Figure 3. The graph G depicted in Figure 3(a) has six separation pairs $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{2,7\}$, $\{3,6\}$, and $\{4,5\}$. The graph G is decomposed into nine split components as shown in Figure 3(b), and into seven 3-connected components $F_1,F_2,\ldots,F_7$ as shown in Figure 3(c). The components F_1,F_2, and F_6 are 3-connected graphs; F_3, F_5, and F_7 are rings; and F_4 is a bond.

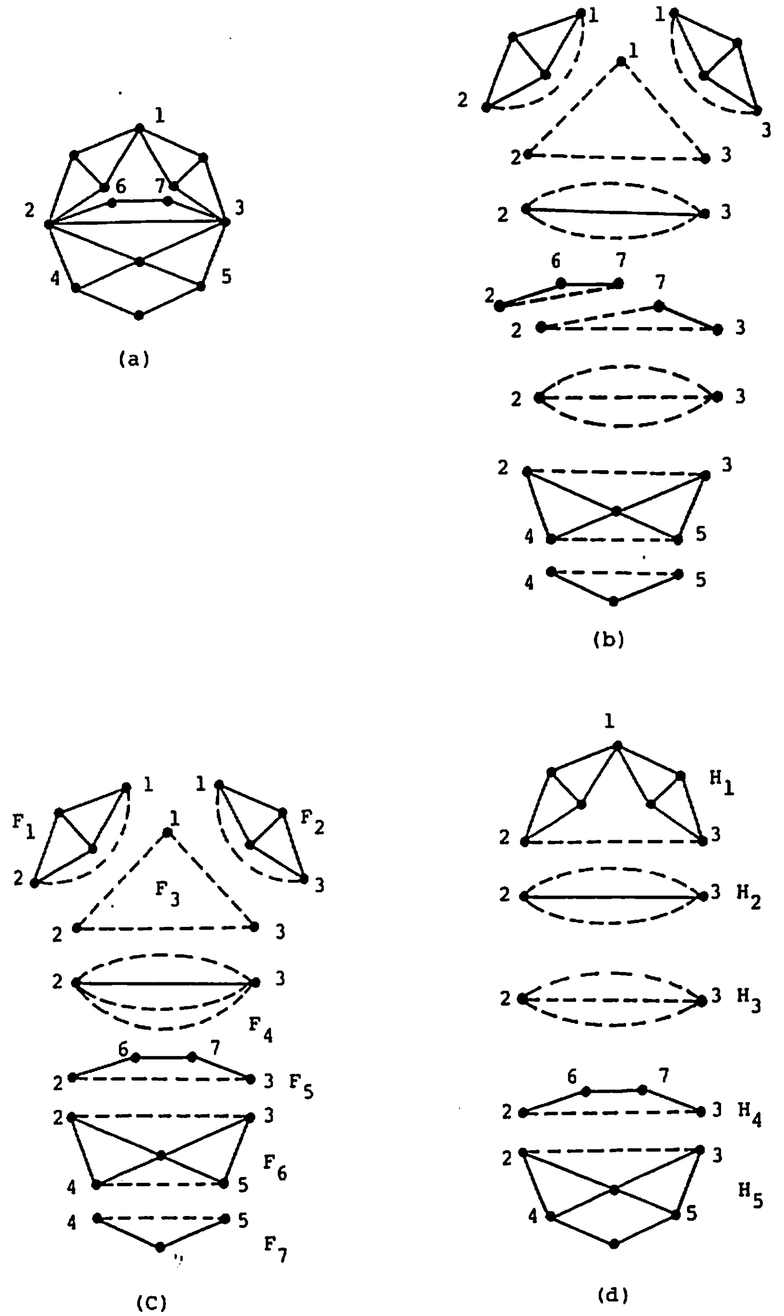


Figure 3. Decompositions of a graph, where virtual edges are written by dashed lines:
(a) a 2-connected graph G;
(b) split components of G;
(c) 3-connected components of G; and
(d) $\{2,3\}$-split components of G with one exception H_3.

Next we introduce new terms. Suppose that $\{x,y\}$ is a separation pair of a graph G and that G is split at $\{x,y\}$, the split graphs are split, and so on, until no more split are possible at $\{x,y\}$ (the remaining graphs are not necessarily 3-connected). A graph constructed in this way is called an *$\{x,y\}$-split component* of G if it has at least one real (i.e. non-virtual) edge. In Figure 3(d), the components H_1, H_2, H_4 and H_5 are the $\{2,3\}$-split components. A separation pair $\{x,y\}$ is *prime* if x and y are the end vertices of a virtual edge contained in a 3-connected component. In other words, a separation pair $\{x,y\}$ is prime if there exist two subgraphs G'_1 and G'_2 such that G'_1 and G'_2 satisfy the conditions (a) and (b) aforementioned and either G'_1 or G'_2 is 2-connected or is a subdivision of an edge joining two vertices of degree three or more. As known from Fig.3(c), separation pairs $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, and $\{4,5\}$ are prime, but $\{2,7\}$ and $\{3,6\}$ are not.

In some cases it can be easily known from the number of $\{x,y\}$-split components or the structures that a graph can never be drawn convex. A *forbidden separation pair* $\{x,y\}$ is a prime separation pair which has either (i) at least four $\{x,y\}$-split components or (ii) three $\{x,y\}$-split components none of which is either a ring or bond. Note that an $\{x,y\}$-split component corresponds to an edge (x,y) of G if it is a bond, and corresponds to a subdivision of an edge (x,y) if it is a ring. The graph in Figure 3(a) has exactly one forbidden separation pair $\{2,3\}$. It will be shown later in Section 4.2 that if a planar graph G has a forbidden separation pair then G can never be drawn convex: G has no extendable facial cycle.

On the other hand the converse of the fact above is not true. In order to say more precisely, we need one more term. A *critical separation* pair $\{x,y\}$ is a prime separation pair which has either (i) three $\{x,y\}$-split components including a ring or a bond, or (ii) two $\{x,y\}$-split components neither of which is a ring. In the graph of Figure 3(a), prime separation pairs $\{1,2\}$ and $\{1,3\}$ are critical, but $\{4,5\}$ is neither forbidden nor critical. When G has no forbidden separation pair, there happen two cases: if G has no critical separation pair either, then G is a subdivision of a 3-connected graph, and so every facial cycle of G is extendable; otherwise, that is, if G has critical separation pairs, a facial cycle S of G may or may not be extendable, depending on the interaction of S and critical separation pairs. (The detailed criterion, called Condition II, will be given in Section 4.2.)

4.2. Condition II

We now give a condition suitable for the testing algorithm, which is equivalent to Tutte's condition [5] or to Condition I under a restriction that S^* is *strict*, that is, every vertex of S is an apex of S^*.

LEMMA 2. Let $G=(V,E)$ be a 2-connected plane graph with the outer facial cycles S, and let S^* be a strict convex polygon of S. Then S^* is extendable if and only if G and S satisfies the following Condition II.

CONDITION II

(a) G has no forbidden separation pair;

(b) For each critical separation pair $\{x,y\}$ of G there exists at most one $\{x,y\}$-split component having no edge of S. Moreover, such an $\{x,y\}$-split component is either a bond if $(x,y)\in E$ or a ring otherwise.

PROOF. We shall show that Condition II is equivalent to Condition I under the restriction that every vertex of S is an apex of S^*.

CONDITION I IMPLIES CONDITION II: Let $\{x,y\}$ be a prime separation pair of G, and let $H_1,H_2,\ldots,H_m$ be the $\{x,y\}$-split components having no edges of S. Then we can show that $m=0$ or 1 and that if $m=1$ then H_1 is neither a ring or a bond, as follows. First suppose that one of the $\{x,y\}$-split components, say H_1, is neither a ring nor a bond, then H_1 has a vertex v $(\neq x,y)$ of degree three or more. Clearly there exist no three paths disjoint except v, each joining v and a vertex of S, since such a path must contain either x or y. This contradicts Condition I(a). Thus every H_i must be either a ring or a bond. Next suppose that $m\geq 2$, then H_1 together with H_2 forms a cycle in G which has no edge in common with S. The cycle has exactly two vertices x and y of degree ≥ 3 in G, contrary to Condition I(c).

Since at most two $\{x,y\}$-split components contain edges of S, there are at most three $\{x,y\}$-split components and moreover one of them is a bond or a ring if there are three. Thus $\{x,y\}$ is not forbidden.

Let $\{x,y\}$ be critical and $m=1$. If $(x,y)\notin E$, then clearly H_1 is not a bond, so H_1 must be a ring. Thus we may assume that $(x,y)\in E$. Suppose that H_1 is a ring. Then edge (x,y) is in S, and so the $x-y$ path in H_1, which is a connected component of $G-V(S)$, is adjacent only with the vertices x and y in S, contradicting Condition I(b). Note that edge (x,y) is a side of S^*. Thus H_1 must be a bond.

CONDITION II IMPLIES CONDITION I: First suppose that G has a vertex v of degree three or more, not satisfying Condition I(a). Then, using Menger's theorem [1, p.47] one can easily show that there exists a prime separation pair $\{x,y\}$ such that one of the $\{x,y\}$-split components H_i contains v and has no edge of S. Since H_i contains a vertex v of degree three or more, H_i is neither a ring nor a bond, contradicting Condition II.

Next suppose that Condition I(b) is not satisfied. Then there exists a connected component C of $G-V(S)$ such that only the vertices x and y of an edge (x,y) on S are adjacent with vertices in C, because G has no multiple edges and S^* is strict. Therefore $\{x,y\}$ is a prime separation pair, and clearly the $\{x,y\}$-split component containing C has no edge in common with S and is not a bond since G is simple. This contradicts Condition II.

Finally suppose that there exists a cycle Z in G violating Condition I(c). Since G is 2-connected, Z has exactly two vertices x and y of degree ≥ 3. Of course $\{x,y\}$ is a prime separation pair. If $(x,y) \in E$, then an $\{x,y\}$-split component having no edges of S is a ring. Otherwise, there are two $\{x,y\}$-split components having no edges of S. Either case contradicts Condition II.

It should be noted that Condition II does not depend on the drawing S^* of S at all. One may suspect that the restriction on S^* loses the generality: there would be an extendable convex polygon of S even if there is no extendable strict S^*. However the following lemma dispels this suspicion.

LEMMA 3. Assume that G is a 2-connected plane graph with the outer facial cycle S, and that S has an extendable convex polygon. Then every strict convex polygon of S is extendable.

PROOF. Immediately follows from either Theorems I and II of [5] or Condition I.

Lemmas 2 and 3 immediately lead to the following theorem.

THEOREM 2. A facial cycles S of a 2-connected *planar* graph G is extendable if and only if S and G satisfies Condition II.

REMARK: Thomassen claims without proof that under the same restriction as ours Condition I is equivalent to condition (d) in [4, p.256] which is similar to Condition II. However one can easily find a counter example to his claim.

4.3 Testing Algorithm.

Condition II is more suitable for testing than Condition I. In this section, we show that the convex testing, i.e. checking Condition II, can be reduced to the planarity testing of a certain graph.

Theorem 2 immediately yields the following corollaries.

COROLLARY 1. A 2-connected planar graph G has no convex drawing if G has a forbidden separation pair.

COROLLARY 2. A 2-connected planar graph G has a convex drawing for any facial cycle of G if G has no forbidden or critical separation pairs.

COROLLARY 3. A 3-connected planar graph G has a convex drawing for any facial cycle of G.

COROLLARY 4. If a facial cycle S of a 2-connected planar graph G satisfies Condition II, then S contains every vertex of critical separation pairs of G.

PROOF. Assume that $\{x,y\}$ is a critical separation pair of G and that S does not contain a vertex x of the pair $\{x,y\}$. Then exactly one $\{x,y\}$-split component contains all the edges of S. On the other hand, Condition II implies that there exists exactly one $\{x,y\}$-split component not containing edges of S, and it must be a ring or a bond. Thus there are exactly two $\{x,y\}$-split components, one of which is a ring or a bond. Then $\{x,y\}$ could not be critical, contrary to the assumption.

We will show in Theorem 3 that the converse of the claim of Corollary 4 is also true in a certain sense. Before presenting Theorem 3 we need the following two lemmas.

LEMMA 4. Suppose that a 2-connected planar graph G has no forbidden separation pair and has exactly one critical separation pair. Then G has a convex drawing.

PROOF. Let $\{x,y\}$ be the critical separation pair. By the definition of a critical separation pair, one can easily observe that G is one of the seven types in Figure 4, in which a shaded part corresponds to an $\{x,y\}$-split component which is neither a ring nor a bond. In each case, one can easily verify that the cycle indicated by a bold line satisfies Condition II (b). Q.E.D.

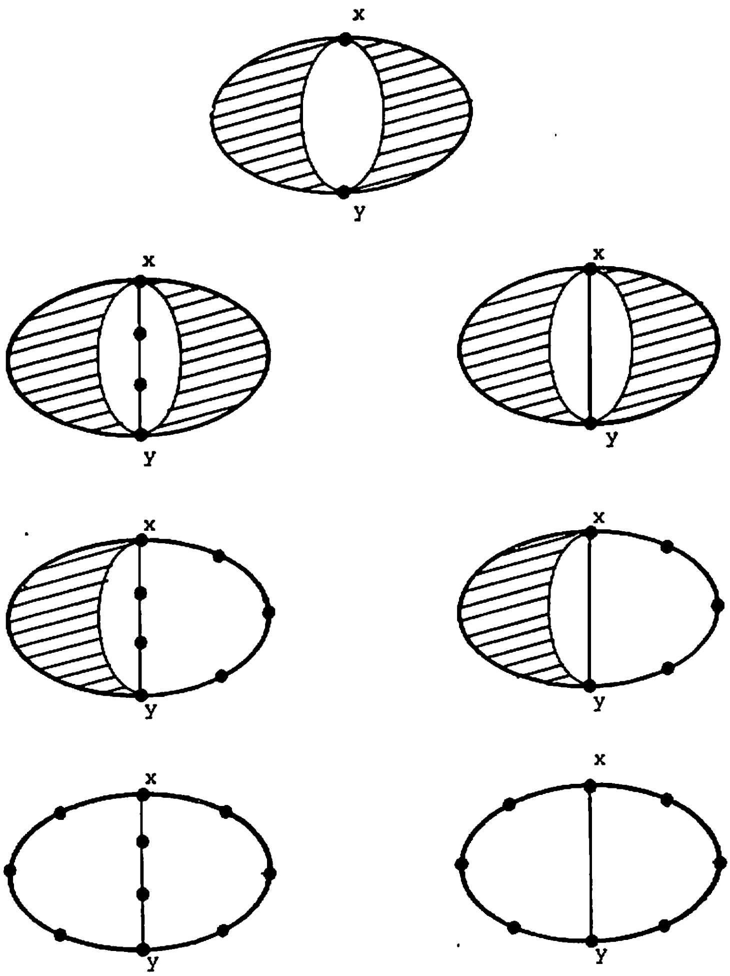


Figure 4. Seven types of a plane graph having exactly one critical separation pair $\{x,y\}$.

Thus we concentrate on a graph having two or more critical separation pairs.

LEMMA 5. Let G be a 2-connected planar graph, and let $\{x,y\}$ be a prime separation pair of G. If a facial cycles S of G contains the vertices x and y, then exactly two $\{x,y\}$-split components contain edges of S.

PROOF. Since S is a cycle, at most two $\{x,y\}$-split components contains edges of S. Furthermore, since S is a facial cycle, not all the edges of S are contained in a single $\{x,y\}$-split component. Thus exactly two $\{x,y\}$-split components contain edges of S.

We are now ready to present Theorem 3 which plays a crucial role in our testing algorithm.

THEOREM 3. Suppose that a 2-connected planar graph G has no forbidden separation pair and has two or more critical separation pairs. Apply the following operation to G for every critical separation pair $\{x,y\}$ of G: if $(x,y) \in E$, then delete edge (x,y) from G; otherwise and if exactly one $\{x,y\}$-split component is a ring, then delete the $x-y$ path in the component from G. Let G_1 be the resulting graph. (Graphs G, and G_1 are illustrated in Figs. 5(a), and (b), respectively.) Then S is an extendable facial cycle of G if and only if S is a facial cycle of G_1 which contains all the vertices of critical separation pairs of G.

PROOF. *Necessity:* Assume that S is the extendable outer cycle of a *plane* graph G. Then, since S satisfies Condition II, all the deleted edges or paths are not on S, and hence S remains to be the outer cycle of the plane subgraph G_1 of G. Moreover S contains all the vertices of the critical separation pairs by Corollary 4.

Sufficiency: Assume that S is a facial cycle of G_1 which contains all the vertices of critical separation pairs of G. Clearly S is also a facial cycle of G. Let $\{x,y\}$ be a critical separation pair of G. Then, by Lemma 5, exactly two $\{x,y\}$-split components, say H_1 and H_2, contain edges of S. Therefore G has at most one $\{x,y\}$-split component containing no edges of S. Suppose that there exists such a component H_3 and that H_3 is neither a ring nor a bond. Then H_3 contains no vertex of critical separation pairs except x and y: if H_3 contains a vertex $z(\neq x,y)$ of a critical separation pair, H_3 would contains an edge of S since S contains all the vertices of critical separation pairs, contrary to the supposition. Therefore there is a critical separation pair $\{u,v\}$, different from $\{x,y\}$, such that vertex u or v is not contained in H_3. Note that G has at least two critical separation pairs. Therefore H_1 or H_2, say H_1, is neither a ring nor a bond, and the other H_2 is a ring or a bond. Then edge (x,y) or the $x-y$ path in H_2 should have been deleted in G_1, so H_2 could not contain edges of S, contrary to the assumption. Hence H_3 must be either a ring or a bond. Thus we have shown that S with G satisfies Condition II (b), so S is extendable.

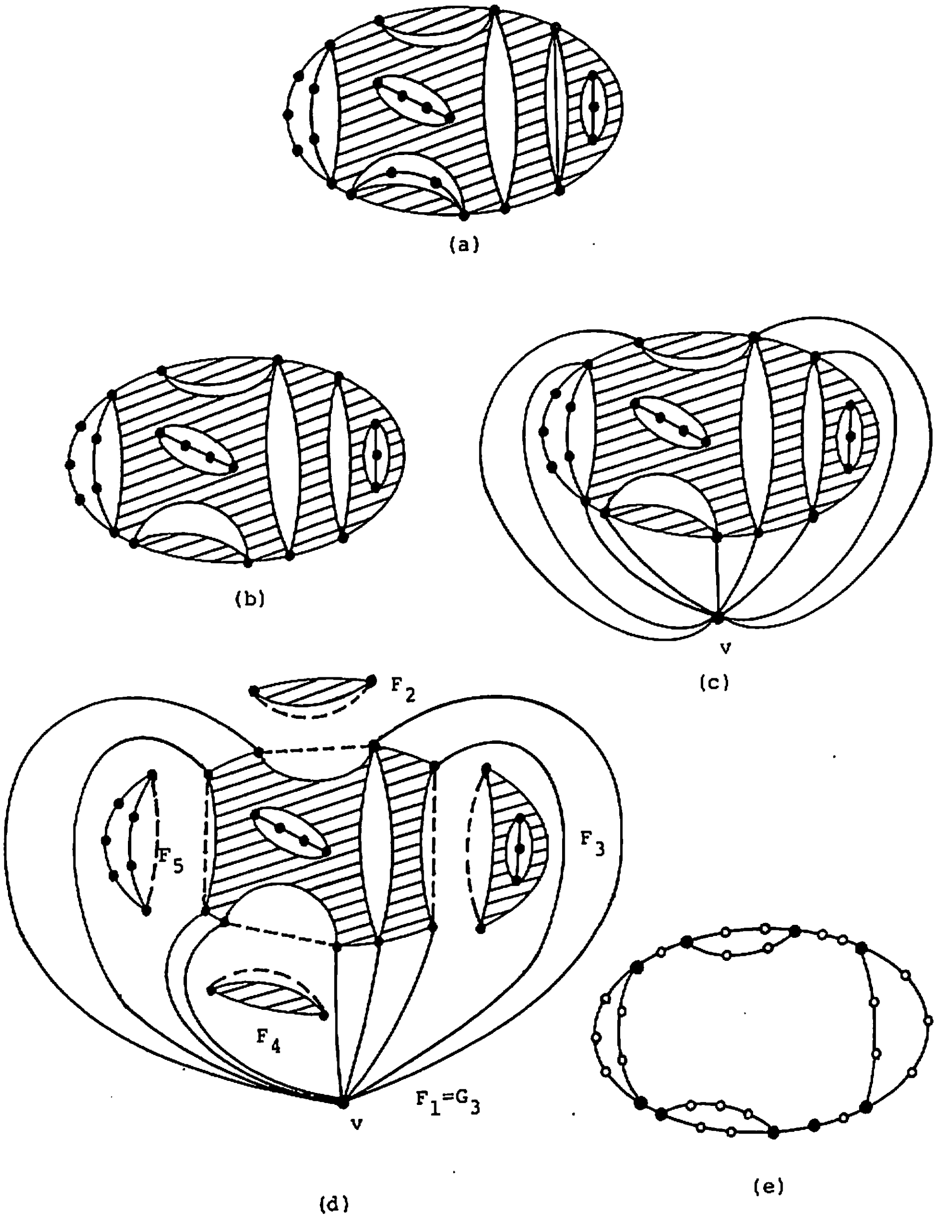


Figure 5. Illustrations of graphs:
(a) G; (b) G_1; (c) G_2;
(d) $F_1 = G_3$; F_2, F_3, and F_4
are in group (b); and F_5 are
in group (c); and
(e) the union of all the extendable facial cycles of G.

We immediately have the following corollaries from Theorem 3. We define here one more term. Let v be a vertex of a 2-connected plane graph G_2 and let $G_1 = G_2 - v$ be 2-connected. Then the *v-cycle* of G_2 is the cycle of the plane subgraph G_1 of G_2 which bounds the face of G_1 in which v lay.

COROLLARY 5. Suppose that a 2-connected planar graph G has no forbidden separation pair and has two or more critical separation pairs. Let G_1 be the graph defined in Theorem 3. Let G_2 be the graph obtained from G_1 by adding a new vertex v and joining v to all the vertices of critical separation pairs of G. (Graph G_2 is illustrated in Figure 5(c).) Then S is an extendable facial cycle of G if and only if

(a) G_2 is planar; and

(b) S is the v-cycle of a plane embedding of G_2.

COROLLARY 6. Suppose that a 2-connected planar graph G has no forbidden separation pair and has two or more critical separation pairs. Let G_2 be the graph defined in Corollary 5. Then G has a convex drawing if and only if G_2 is planar.

Combining Corollaries 1, 2, 5, 6, and Lemma 4, we immediately have the following linear testing algorithm.

Algorithm CONVEX-TESTING

STEP 1. Finds all the separation pairs of the given 2-connected planar graph G by the linear algorithm of Hopcroft and Tarjan [2] for finding 3-connected components. Determine three sets of separation pairs: the sets PSP of prime separation pairs; the set CSP of critical separation pairs; and the set FSP of forbidden separation pairs. If FSP$\neq\emptyset$ then stop with a message "G has no convex drawing" (see Corollary 1). If FSP$=$CSP$=\emptyset$ then print a message "All facial cycles are extendable" (see Corollary 2), and go to Step 3. If $|\text{CSP}|=1$ then let S be an extendable outer facial cycle depicted in Figure 4, and go to Step 3 (see Lemma 4).

STEP 2. Obtain a graph G_1 from G by applying the operation in Theorem 3 to every critical separation pair of G. Construct a graph G_2 from G_1 by adding a new vertex v and joining v to all the vertices of CSP. Test the planarity of G_2 by the linear algorithm of Hopcroft and Tarjan [3] (see Corollary 6). If G is nonplanar, then stop with a message "G has no convex drawing (There is no facial cycles containing every vertex of CSP)." Otherwise let S be the v-cycle of a plane graph G_2. Necessarily S satisfies Condition II (see Corollary 5).

STEP 3. Stop with a message "The given graph G has a convex drawing."

4.4 Finding all Extendable Facial Cycles.

Algorithm CONVEX-TESTING in Section 4.3 finds an extendable facial cycle S if any. In this section we show how to generate all the extendable facial cycles of G. If the given graph G has exactly one critical separation pair, then one can easily enumerate all the extendable facial cycles, since G has at most four such facial cycles, as verified by checking the seven types in Figure 4. Thus we may assume that a graph G has two or more critical separation pairs.

Suppose that the outer facial cycle S of a 2-connected plane graph G is extendable. Let G_2 be the graph defined in Corollary 5. Consider how to generate all the v-cycles of plane embeddings of G_2. The 3-connected components of G_2 are classified into five groups:

(1) the 3-connected graph containing the vertex v;

(2) 3-connected graphs containing the vertices of exactly one critical separation pair of G (Only the graph in (1) can contain vertices of two or more critical separation pairs);

(3) rings containing the vertices of noncritical separation pair G (each of these corresponds to a subdivision of an edge);

(4) rings containing the vertices of a critical separation pair $\{x,y\}$ of G (if there exists exactly one such ring then an $x-y$ path corresponding to the ring should have been deleted in G_1, so there must exist exactly two such rings); and

(5) triple bonds containing the vertices of a critical separation pair.

We next merge some of these 3-connected components into graphs which are necessarily subdivisions of 3-connected graphs. The resulting graphs are classified into three groups, depending on the types of 3-connected components involved in the merge:

(a) the graph G_3 obtained from the 3-connected graph of (1) and 3-connected components of type (3);

(b) graphs obtained from 3-connected components of type (2) or (3); and

(c) graphs obtained from 3-connected components of type (4) or (5).

The resulting graphs are illustrated in Figure 5(d), in which F_1 is G_3, F_2, F_3, and F_4 are in group (b), and F_5 is in group (c).

We now consider the embeddings of G_2. Since a 3-connected planar graph and its subdivision graph are uniquely embeddable [1], the graph G_3 has the unique plane embedding. Now consider a planar graph G'_3 obtained by merging G_3 and one of the graphs in group (b) or (c), say F. Since every graph in group (b) or (c) does not contain a critical separation pair, that is, is a subdivision of a 3-connected graph, G'_3 has exactly two distinct plane embeddings: one is obtained by merging the plane graph G_3 and a plane graph F; and the other is obtained from it by replacing F with its mirror image. Thus, if there exist k graphs in groups (b) and (c), the planar graph G_2 has 2^k distinct plane embeddings, from each of which a distinct v-cycle of G_2 arises. Not that all the graphs in groups (b) and (c) are, of course, disjoint.

Thus we can find all the extendable facial cycles. However, a graph may have an exponential number of these facial cycles, so we do not enumerate these facial cycles. We simply represent these cycles by the union of them, from which one can easily pick up whichever one likes. As an example, we illustrate in Figure 5(e) the representation of all the extendable facial cycles of G in Figure 5(a), in which every cycle containing all the "black" vertices of the critical separation pairs of G is extendable. Replace Step 3 of the algorithm CONVEX-TESTING by the following Steps 3 and 4, then the revised algorithm constructs the representation of all the extendable facial cycles of a given graph G.

STEP 3. If $|CSP|=1$ then enumerate all the extendable facial cycles, and go to Step 4. Otherwise, construct the union of all the extendable facial cycles of G as follows. Divide G_2 into the 3-connected components and merge these 3-connected components into graphs of the three groups (a), (b) and (c) above. For each graph F in group (b) or (c) apply the following procedure: let (x,y) be the virtual edge in F, and let P_{xy} be the $x-y$ path on S joining x and y in F; let F' be the plane graph obtained from a plane graph F by deleting the virtual edge (x,y); find another $x-y$ path Q_{xy} in F such that $P_{xy} \cup Q_{xy}$ is a facial cycle of F'; and add Q_{xy} to S by joining them at the vertices x and y. (Clearly the cycle S' obtained from S by replacing P_{xy} by Q_{xy} is one of the v-cycles of G_2, and so S' is extendable.)

STEP 4. Stop with a message "All facial cycles have been found."

We have the following result on the above algorithm.

THEOREM 4. The revised CONVEX-TESTING determines whether a 2-connected planar graph has a convex drawing, and moreover finds all the extendable facial cycles. It uses linear time and space.

Finally we remark that one can easily find all extendable convex polygons of an extendable facial cycles (left to the reader).

5. Example.

The algorithms CONVEX-TESTING and CONVEX-DRAWING have been implemented in PASCAL and run on a small computer FACOM 230/38s. Experiments indicate that the algorithm can find all the extendable facial cycles of a graph having 150 edges in less than 6 seconds. We illustrate a computational example in Figure 6. Fig. 6 (a) depicts an input graph G having 50 vertices and 83 edges. Algorithm CONVEX-TESTING finds all the extendable facial cycles of G, the union of which is represented as shown in Figure 6 (b). A pair of paths, either of which can be a part of an extendable facial cycle, are in parentheses. Since there are two pairs of such paths, Figure 6(b) represents the four extendable facial cycles:

$$C_1=15-16-1-2-3-4-5-6-7-8-9-10-11-12-13-14$$

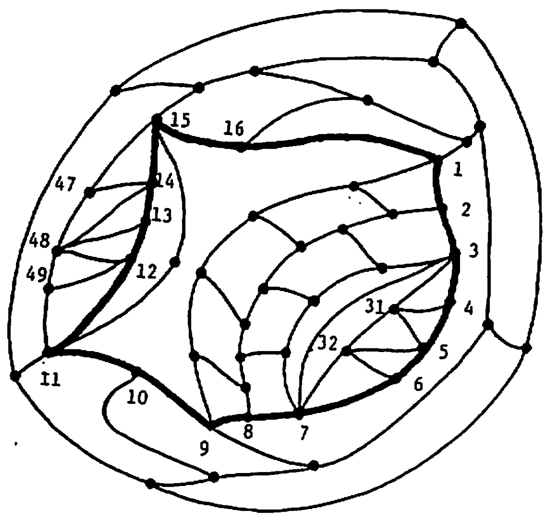
$$C_2=15-16-1-2-3-4-5-6-7-8-9-10-11-49-48-47$$

$$C_3=15-16-1-2-3-31-32-7-8-9-10-11-12-13-14$$

$$C_4=15-16-1-2-3-31-32-7-8-9-10-11-49-48-47.$$

The cycle C_1 is indicated by bold lines in Figure 6(a). When both a plane graph G with the outer cycle C_1 and a convex polygon (regular 16-gon) of C_1 are given, algorithm CONVEX-DRAWING obtains a convex drawing of G, which is depicted in Figure 6(c).

(a)



(b) THE EXTENDABLE FACIAL CYCLES OF GIVEN GRAPH
15 16 1 2 3 (4 5 6)(31 32) 7 8 9 10 11 (12 13 14)(49 48 47)

(c)

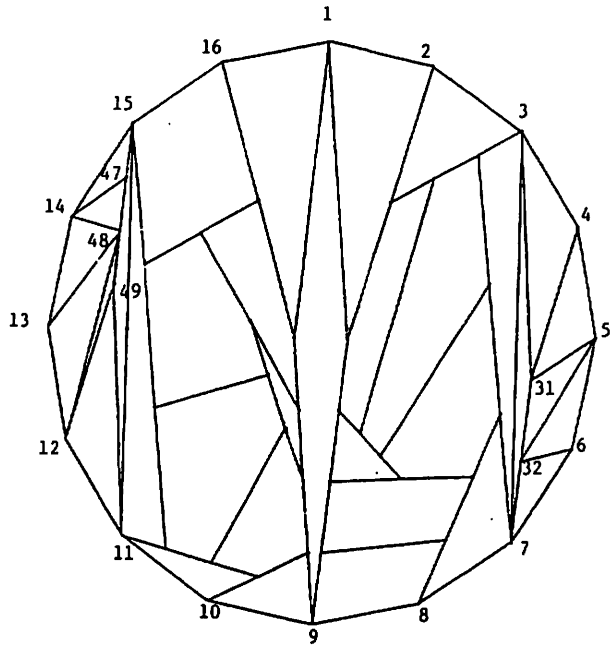


Figure 6. A computational example: (a) input graph G (some vertex numbers are not shown for simplicity); (b) output of CONVEX-TESTING; and (c) output of CONVEX-DRAWING.

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