

A Linear Algorithm for Edge-Coloring Series–Parallel Multigraphs

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Many combinatorial problems can be efficiently solved for series–parallel multigraphs. However, the edge-coloring problem of finding the minimum number of colors required for edge-coloring given graphs is one of a few well-known combinatorial problems for which no efficient algorithms have been obtained for series–parallel multigraphs. This paper gives a linear algorithm for the problem on series–parallel multigraphs. © 1996 Academic Press, Inc.

1. INTRODUCTION

This paper deals with the edge-coloring problem to find the minimum number of colors required for edge-coloring a given graph G so that no two adjacent edges are colored with the same color. The minimum number is called the *chromatic index* of G , and denoted by $\chi'(G)$. It is known that many combinatorial problems can be solved very efficiently, say in linear time, for series–parallel graphs or partial k -trees [1, 2, 4, 5, 9]. Such a class of problems has been characterized in terms of “forbidden graphs” or “extended monadic logic of second order” [1, 2, 4, 5, 9]. The edge-coloring problem does not belong to such a class of “maximum (or minimum) subgraph problems,” and is indeed one of the “edge-covering problems” which, as mentioned in [4], do not appear to be solvable in linear time for partial k -trees. Furthermore the edge-coloring problem is one of a few well-known problems for which no efficient algorithms of low complexity order have been obtained even for series–parallel multigraphs. For series–parallel *simple* graphs, which have no multiple edges, the edge-coloring problem is known to be solvable in time $O(n^2)$, where n is the number of vertices in a graph [10]. Sysłó, referring to [10], conjectured that there existed a linear algorithm to solve the edge-coloring problem for series-parallel *multigraphs*, which may have multiple edges [8]. Very re-

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cently, Bodlaender gave a polynomial-time algorithm of high complexity order to solve the edge-coloring problem for series-parallel (simple) graphs or partial k -trees [3]. His algorithm runs in time $O(n\Delta^{64})$ for series-parallel multigraphs. Note that the maximum degree Δ is not bounded in general. On the other hand, recently Seymour [7] and Marcotte [6] have shown that a kind of a “min-max formula” holds for the chromatic index $\chi'(G)$ of a series-parallel multigraph G and a trivial lower bound on $\chi'(G)$. Using the formula, one can find the chromatic index $\chi'(G)$ of G in polynomial time, say in time $O(n\Delta^3)$.

In this paper we give a linear algorithm to find the chromatic index $\chi'(G)$ for any series-parallel multigraph G . Thus the complexity is greatly improved. Our algorithm is not based on the min-max formula above, but uses a number of new ideas specific to series-parallel multigraphs as well as a general technique employed by Bodlaender for partial k -trees. Our algorithms for series-parallel multigraphs can easily be extended to the case of partial 2-trees. Note that every 2-connected component of a partial 2-tree is a series-parallel simple graph.

The paper is organized as follows. Section 2 includes basic definitions and notations. Section 3 gives a simple algorithm which computes $\chi'(G)$ of a given series-parallel multigraph G in time $O(n\Delta^3)$. Section 4 improves the time complexity to linear time. Finally we conclude in Section 5 with a discussion of the results and some related works. Early versions of this paper were presented in [13, 14].

2. DEFINITIONS

In this section we give some basic definitions. We denote by $G = (V, E)$ a graph with vertex set V and edge set E . The set of vertices and the set of edges of G are often denoted by $V(G)$ and $E(G)$, respectively. A (two-terminal) *series-parallel multigraph* is defined recursively as follows:

1. A graph G of a single edge is a series-parallel multigraph. The ends v_s and v_t of the edge are called the *terminals* of G and denoted by $v_s(G)$ and $v_t(G)$.
2. Let G_1 be a series-parallel multigraph with terminals $v_s(G_1)$ and $v_t(G_1)$, and let G_2 be a series-parallel multigraph with terminals $v_s(G_2)$ and $v_t(G_2)$.

(a) A graph G obtained from G_1 and G_2 by identifying vertex $v_t(G_1)$ with vertex $v_s(G_2)$ is a series-parallel multigraph whose terminals are $v_s(G) = v_s(G_1)$ and $v_t(G) = v_t(G_2)$. Such a connection is called a *series connection*, and G is denoted by $G = G_1 \cdot G_2$.

(b) A graph G obtained from G_1 and G_2 by identifying $v_s(G_1)$ with $v_s(G_2)$ and $v_t(G_1)$ with $v_t(G_2)$ is a series-parallel multigraph whose terminals are $v_s(G) = v_s(G_1) = v_s(G_2)$ and $v_t(G) = v_t(G_1) = v_t(G_2)$. Such a connection is called a *parallel connection*, and G is denoted by $G = G_1 \parallel G_2$. (See Fig. 1.)

A series-parallel multigraph G can be represented by a “binary decomposition tree” [9]. Figure 2 illustrates a series-parallel multigraph G and its binary decomposition tree T_b . Labels s and p attached to internal nodes in T_b indicate series and parallel connections, respectively, and nodes labeled s and p are called *s-* and *p-nodes*, respectively. A node u of tree T_b corresponds to a subgraph of G , which is denoted by G_u . Especially, every leaf of T_b represents a subgraph of G induced by two vertices, that is, a set of multiple edges. The set of multiple edges joining vertices u and v is denoted by $E(u, v)$.

The terminals $v_s(G)$ and $v_t(G)$ of G are often denoted simply by v_s and v_t . We denote by $d(v)$ the degree of vertex $v \in V$ in G . The maximum degree of G is denoted by Δ . Let $d(G) = d(v_s) + d(v_t)$, $\delta_{st}(G) = \min\{d(v_s), d(v_t)\}$, and $\Delta_{st}(G) = \max\{d(v_s), d(v_t)\}$.

An *edge-coloring* of G is an assignment of colors to the edges in G such that no two adjacent edges have the same color. Figure 3 depicts an edge-coloring of a graph G in Fig. 2a. The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors used by any edge-coloring of G . The number of colors used by an edge-coloring φ is denoted by $\#\varphi$, the color assigned to edge $e \in E$ is denoted by $\varphi(e)$, and for a set $A \subseteq E$ of edges, $\varphi(A)$ denotes the set of colors assigned to the edges in A . Denote by $\varphi(v)$ the set of colors assigned to the edges incident with vertex $v \in V$. Let $S(\varphi) = \varphi(v_s) - \varphi(v_t)$, $T(\varphi) = \varphi(v_t) - \varphi(v_s)$, $R(\varphi) = \varphi(v_s) \cap \varphi(v_t)$, and $Q(\varphi) = \varphi(E) - (\varphi(v_s) \cup \varphi(v_t))$. Let $s(\varphi) = |S(\varphi)|$, $t(\varphi) = |T(\varphi)|$, $r(\varphi) = |R(\varphi)|$, and $q(\varphi) = |Q(\varphi)|$. These symbols are illustrated in Fig. 4.

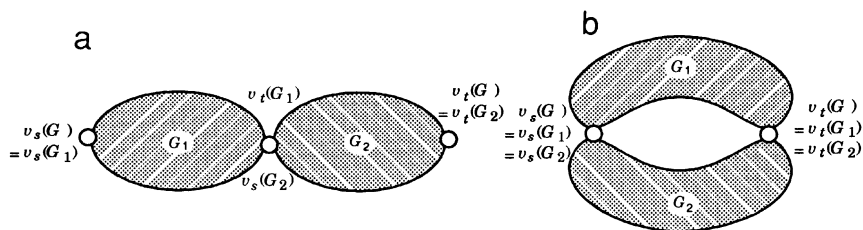


FIG. 1. Series (a) and parallel (b) connections.

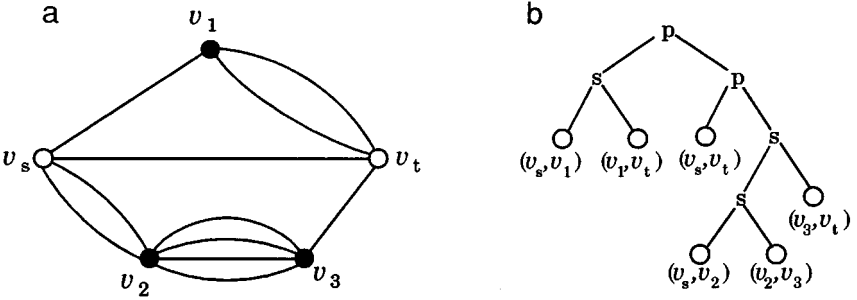


FIG. 2. (a) A series-parallel graph G and (b) its binary decomposition tree T_b .

Clearly the following equations hold:

$$\begin{aligned} \# \varphi &= s(\varphi) + t(\varphi) + r(\varphi) + q(\varphi); \\ d(v_s) &= s(\varphi) + r(\varphi); \\ d(v_t) &= t(\varphi) + r(\varphi); \quad \text{and} \\ d(G) &= s(\varphi) + t(\varphi) + 2r(\varphi). \end{aligned}$$

The edge-coloring φ of a graph G in Fig. 3 satisfies $s(\varphi) = 1$, $t(\varphi) = 1$, $r(\varphi) = 3$, and $q(\varphi) = 1$.

One can observe that $\chi'(G)$ of a series-parallel multigraph G cannot be computed directly from $\chi'(G_1)$ and $\chi'(G_2)$ when $G = G_1 \parallel G_2$, although $\chi'(G)$ can be computed directly from $\chi'(G_1)$ and $\chi'(G_2)$ when $G = G_1 \cdot G_2$ (see Lemma 4.4(c)). So we introduce a new invariant $\chi'(G, i)$ of G to compute $\chi'(G)$ by a dynamic programming algorithm. For every nonnegative integer i , $\chi'(G, i)$ is defined to be the minimum number of

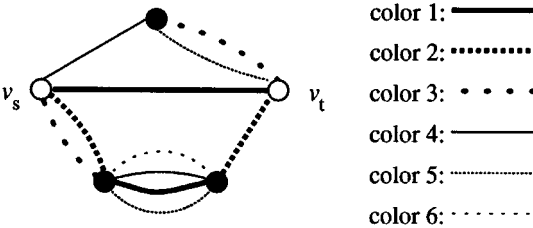


FIG. 3. Edge-coloring φ with $\# \varphi = 6$ and $r(\varphi) = 3$.

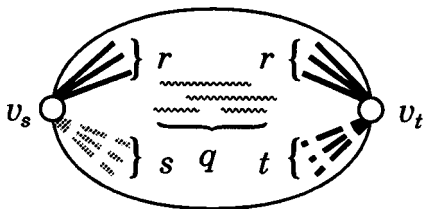


FIG. 4. Illustration for symbols.

colors used by an edge-coloring of G such that exactly i common colors appear at both v_s and v_t ; that is,

$$\chi'(G, i) = \begin{cases} \min\{\#\varphi \mid \varphi \text{ is an edge-coloring of } G \text{ with } r(\varphi) = i\} \\ \text{if } G \text{ has such a coloring;} \\ \infty \quad \text{otherwise.} \end{cases}$$

Define $i_{\min}(G)$ and $i_{\max}(G)$ as follows:

$$i_{\min}(G) = \min\{i \mid G \text{ has a coloring } \varphi \text{ with } r(\varphi) = i\}, \quad \text{and}$$

$$i_{\max}(G) = \max\{i \mid G \text{ has a coloring } \varphi \text{ with } r(\varphi) = i\}.$$

Any coloring φ of G satisfies $r(\varphi) \geq |E(v_s, v_t)|$, and a trivial coloring φ assigning a distinct color to every edge satisfies $r(\varphi) = |E(v_s, v_t)|$. Therefore $i_{\min}(G) = |E(v_s, v_t)|$. Clearly $i_{\max}(G) \leq \delta_{st}(G)$ and

$$\chi'(G) = \min\{\chi'(G, i) \mid i_{\min}(G) \leq i \leq i_{\max}(G)\}.$$

For the graph G depicted in Fig. 2a, $\chi'(G, 0) = \infty$, $\chi'(G, 1) = 7$, $\chi'(G, 2) = 6$, $\chi'(G, 3) = 6$, $\chi'(G, 4) = 7$, $\chi'(G, 5) = \infty$, $i_{\min}(G) = 1$, $i_{\max}(G) = 4$, and consequently $\chi'(G) = 6$.

For an edge-coloring φ of a series-parallel multigraph G such that $G = G_1 \parallel G_2$ or $G = G_1 \cdot G_2$, let φ_1 and φ_2 be restrictions of φ to $E(G_1)$ and $E(G_2)$, respectively; that is, φ_1 is an edge-coloring of G_1 such that $\varphi_1(e) = \varphi(e)$ for $e \in E(G_1)$ and φ_2 is an edge-coloring of G_2 such that $\varphi_2(e) = \varphi(e)$ for $e \in E(G_2)$. We denote by $d_i(v)$ the degree of vertex v in G_i , $i = 1$ or 2 .

3. SIMPLE ALGORITHM

In this section we present a simple algorithm to solve the edge-coloring problem for series-parallel multigraphs G , although it is not a linear algorithm. Our idea is to compute $\chi'(G, i)$ recursively for all integers i ,

$i_{\min}(G) \leq i \leq i_{\max}(G)$. We will show in Lemmas 3.3 and 3.5 that if $G = G_1 \parallel G_2$ or $G = G_1 \cdot G_2$ then $\chi'(G, i)$ for $i, i_{\min}(G) \leq i \leq i_{\max}(G)$, together with $i_{\min}(G)$ and $i_{\max}(G)$, can be computed from the counterparts of G_1 and G_2 . We first have the following lemma.

LEMMA 3.1. *Let G be a series-parallel multigraph. Then*

- (a) $\chi'(G, i) \leq \chi'(G, i + 1) + 1$ if $i \geq i_{\min}(G)$, and
- (b) $\chi'(G, i) \neq \infty$ for every $i, i_{\min}(G) \leq i \leq i_{\max}(G)$.

Proof. (a) If $\chi'(G, i + 1) = \infty$, then the inequality trivially holds. Therefore one may assume that $\chi'(G, i + 1) \neq \infty$. Let φ be a coloring of G such that $\#\varphi = \chi'(G, i + 1)$ and $r(\varphi) = i + 1$. Since $r(\varphi) - |E(v_s, v_t)| = (i + 1) - i_{\min}(G) \geq 1$, there is an edge whose color is in $R(\varphi)$ and which is incident with v_s but not incident with v_t . By assigning a new color to the edge one can construct a coloring φ' of G such that $\#\varphi' = \#\varphi + 1 = \chi'(G, i + 1) + 1$ and $r(\varphi') = r(\varphi) - 1 = i$. Therefore $\chi'(G, i) \leq \chi'(G, i + 1) + 1$.

(b) The claim immediately follows from (a). Q.E.D.

Using Lemma 3.1, we have the following lemma.

LEMMA 3.2. *Let $G = G_1 \parallel G_2$, and let i_1 and i_2 be integers such that $i_{\min}(G_1) \leq i_1 \leq i_{\max}(G_1)$ and $i_{\min}(G_2) \leq i_2 \leq i_{\max}(G_2)$. Let*

$$h_p(i_1, i_2) = \min\{\delta_{\text{st}}(G), d(G_1) - i_1 + i_2, d(G_2) - i_2 + i_1\}, \quad \text{and}$$

$$\alpha_p(i, i_1, i_2) = \max\{\chi'(G_1, i_1), \chi'(G_2, i_2), d(G) - i\}.$$

Then G has a coloring φ such that $r(\varphi) = i$, $r(\varphi_1) = i_1$ and $r(\varphi_2) = i_2$ if and only if $i_1 + i_2 \leq i \leq h_p(i_1, i_2)$. Furthermore any such coloring φ satisfies $\#\varphi \geq \alpha_p(i, i_1, i_2)$, and indeed G has such a coloring φ with $\#\varphi = \alpha_p(i, i_1, i_2)$.

Proof. (a) *Necessity.* Let φ be any coloring of G such that $r(\varphi) = i$, $r(\varphi_1) = i_1$, and $r(\varphi_2) = i_2$. Then clearly

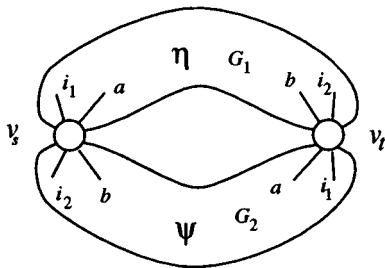
$$\#\varphi \geq \max\{\chi'(G_1, i_1), \chi'(G_2, i_2), d(G) - i\} = \alpha_p(i, i_1, i_2).$$

$R(\varphi)$ is a union of four disjoint subsets as follows (see Fig. 5):

$$R(\varphi) = R(\varphi_1) \cup R(\varphi_2) \cup A \cup B;$$

$$A = S(\varphi_1) \cap T(\varphi_2); \quad \text{and}$$

$$B = T(\varphi_1) \cap S(\varphi_2).$$

FIG. 5. Illustration for symbols a and b .

Obviously

$$|A| \leq \min\{d_1(v_s) - i_1, d_2(v_t) - i_2\} \quad \text{and}$$

$$|B| \leq \min\{d_1(v_t) - i_1, d_2(v_s) - i_2\}.$$

Therefore we have

$$\begin{aligned} i_1 + i_2 \leq i \leq i_1 + i_2 + \min\{d_1(v_s) - i_1, d_2(v_t) - i_2\} \\ + \min\{d_1(v_t) - i_1, d_2(v_s) - i_2\}. \end{aligned}$$

One can easily observe

$$\begin{aligned} h_p(i_1, i_2) = i_1 + i_2 + \min\{d_1(v_s) - i_1, d_2(v_t) - i_2\} \\ + \min\{d_1(v_t) - i_1, d_2(v_s) - i_2\}. \end{aligned}$$

Hence $i_1 + i_2 \leq i \leq h_p(i_1, i_2)$.

(b) *Sufficiency.* Let i be an arbitrary integer such that $i_1 + i_2 \leq i \leq h_p(i_1, i_2)$. We shall construct a coloring φ of G such that $r(\varphi) = i$, $r(\varphi_1) = i_1$, $r(\varphi_2) = i_2$, and $\#\varphi = \alpha_p(i, i_1, i_2)$. By Lemma 3.1(b), G_1 has a coloring η such that $\#\eta = \chi'(G_1, i_1)$ and $r(\eta) = i_1$. Similarly G_2 has a coloring ψ such that $\#\psi = \chi'(G_2, i_2)$ and $r(\psi) = i_2$. One may assume w.l.o.g. that the colors used by η are different from those by ψ , i.e., $\eta(E(G_1)) \cap \psi(E(G_2)) = \emptyset$. Consider an edge-coloring φ of G which is an extension of the two colorings η and ψ , that is,

$$\varphi(e) = \begin{cases} \eta(e) & \text{if } e \in E(G_1); \\ \psi(e) & \text{if } e \in E(G_2). \end{cases}$$

Then $r(\varphi_1) = i_1$ and $r(\varphi_2) = i_2$, but $r(\varphi) = i_1 + i_2$ and $\#\varphi = \chi'(G_1, i_1) + \chi'(G_2, i_2)$. Therefore we shall alter the coloring φ : we shall increase

$r(\varphi)$ from $i_1 + i_2$ to i and decrease $\#\varphi$ from $\chi'(G_1, i_1) + \chi'(G_2, i_2)$ to $\alpha_p(i, i_1, i_2)$.

We first increase $r(\varphi)$ to i by renaming $i - (i_1 + i_2)$ of ψ 's colors. Let

$$a = \min\{i - i_1 - i_2, d_1(v_s) - i_1, d_2(v_t) - i_2\}, \quad \text{and} \\ b = i - i_1 - i_2 - a.$$

Then we have

$$i = i_1 + i_2 + a + b, \\ 0 \leq a \leq \min\{d_1(v_s) - i_1, d_2(v_t) - i_2\}, \quad \text{and} \\ 0 \leq b \leq \min\{d_1(v_t) - i_1, d_2(v_s) - i_2\}.$$

We rename $a + b$ colors of the coloring ψ : replace any a colors in $T(\psi)$ with a colors in $S(\eta)$, and replace any b colors in $S(\psi)$ with b colors in $T(\eta)$. Then $|S(\eta) \cap T(\psi)| = a$, $|T(\eta) \cap S(\psi)| = b$, and hence the coloring φ extended from η and this altered ψ satisfies $r(\varphi) = i_1 + i_2 + a + b = i$.

We finally decrease $\#\varphi$ to $\alpha_p(i, i_1, i_2) = \max\{\chi'(G_1, i_1), \chi'(G_2, i_2), d(G) - i\}$ without changing $r(\varphi)$. One may assume w.l.o.g. that $\chi'(G_1, i_1) \geq \chi'(G_2, i_2)$. Then it suffices to consider the following two cases.

Case 1. $\chi'(G_1, i_1) \geq d(G) - i$. In this case $\alpha_p(i, i_1, i_2) = \chi'(G_1, i_1)$, and consequently we alter ψ so that $\psi(E(G_2)) \subseteq \eta(E(G_1))$ and hence $\#\varphi = \chi'(G_1, i_1)$. Clearly the following four equations hold:

$$d(G_1) = 2i_1 + s(\eta) + t(\eta), \\ d(G_2) = 2i_2 + s(\psi) + t(\psi), \\ d(G) = d(G_1) + d(G_2), \quad \text{and} \\ \chi'(G_1, i_1) = i_1 + s(\eta) + t(\eta) + q(\eta).$$

These four equations together with the two equations $\chi'(G_1, i_1) \geq d(G) - i$ and $i = i_1 + i_2 + a + b$ imply

$$q(\eta) \geq i_2 + (s(\psi) - b) + (t(\psi) - a).$$

Therefore we can rename some more colors of ψ as follows: replace all the i_2 colors in $R(\psi)$, the remaining $s(\psi) - b$ colors in $S(\psi)$, and the remaining $t(\psi) - a$ colors in $T(\psi)$ with colors in $Q(\eta)$. We finally replace all the colors in $Q(\psi)$ with colors in G_1 which have not been used yet for G_2 . This can be done since $\chi'(G_1, i_1) \geq \chi'(G_2, i_2)$. Then the coloring φ

extended from η and this altered ψ satisfies $\# \varphi = \chi'(G_1, i_1)$, $r(\varphi) = i$, $r(\varphi_1) = i_1$, and $r(\varphi_2) = i_2$.

Case 2. $\chi'(G_1, i_1) < d(G) - i$. In this case $\alpha_p(i, i_1, i_2) = d(G) - i$, and consequently we alter η and ψ so that η and ψ , and hence φ , use only the $d(G) - i$ colors in $S(\eta) \cup T(\eta) \cup R(\eta) \cup S(\psi) \cup T(\psi) \cup R(\psi)$, as follows. Since $d(G) - i > \chi'(G_1, i_1)$, we have $q(\eta) < i_2 + (s(\psi) - b) + (t(\psi) - a)$ similarly as above. Therefore we can replace all the colors in $Q(\eta)$ with any $q(\eta)$ colors among the i_2 colors in $R(\psi)$ plus the remaining $s(\psi) - b$ colors in $S(\psi)$ plus the remaining $t(\psi) - a$ colors in $T(\psi)$. Since $d(G) - i > \chi'(G_2, i_2)$, $q(\psi) < i_1 + (t(\eta) - b) + (s(\eta) - a)$. Therefore we can replace all the colors in $Q(\psi)$ with any $q(\psi)$ colors among the i_1 colors in $R(\eta)$ plus the remaining $s(\eta) - a$ colors in $S(\eta)$ plus the remaining $t(\eta) - b$ colors in $T(\eta)$. Then the coloring φ extended from these altered η and ψ satisfies $\# \varphi = d(G) - i$, $r(\varphi) = i$, $r(\varphi_1) = i_1$, and $r(\varphi_2) = i_2$. Q.E.D.

Lemma 3.2 immediately implies the following lemma.

LEMMA 3.3. *Let $G = G_1 \parallel G_2$; then the following hold:*

- (a) $i_{\min}(G) = i_{\min}(G_1) + i_{\min}(G_2)$;
- (b) $i_{\max}(G) = \min\{\delta_{\text{st}}(G), d(G_1) - i_{\min}(G_1) + i_{\max}(G_2), d(G_2) - i_{\min}(G_2) + i_{\max}(G_1)\}$; and
- (c) for every i , $i_{\min}(G) \leq i \leq i_{\max}(G)$,

$$\chi'(G, i) = \min_{i_1, i_2} \alpha_p(i, i_1, i_2),$$

where the minimum is taken over all i_1 and i_2 such that

$$\begin{aligned} i_{\min}(G_1) &\leq i_1 \leq i_{\max}(G_1), \\ i_{\min}(G_2) &\leq i_2 \leq i_{\max}(G_2), \quad \text{and} \\ i_1 + i_2 &\leq i \leq h_p(i_1, i_2). \end{aligned}$$

Proof. (a) Since $i_{\min}(G) = |E(v_s, v_t)|$, the equation clearly holds.

(b) By Lemma 3.2 $i_{\max}(G) = \max_{i_1, i_2} h_p(i_1, i_2)$, where i_1 and i_2 run over all integers such that $i_{\min}(G_1) \leq i_1 \leq i_{\max}(G_1)$ and $i_{\min}(G_2) \leq i_2 \leq i_{\max}(G_2)$. Therefore the claimed equation holds.

(c) The claimed equation immediately follows from Lemma 3.2. Q.E.D.

Similarly we have the following two lemmas for the case $G = G_1 \cdot G_2$.

LEMMA 3.4. *Let $G = G_1 \cdot G_2$, $v = v_t(G_1) = v_s(G_2)$, and let i_1 and i_2 be integers such that $i_{\min}(G_1) \leq i_1 \leq i_{\max}(G_1)$ and $i_{\min}(G_2) \leq i_2 \leq i_{\max}(G_2)$. Let*

$$\begin{aligned} h_s(i_1, i_2) &= \min\{\delta_{st}(G), d(G) - i_1 - i_2\}, \quad \text{and} \\ \alpha_s(i, i_1, i_2) &= \max\{\chi'(G_1, i_1), \chi'(G_2, i_2), \\ &\quad d(v), d(G) - i, d(v) - i_1 - i_2 + i, \\ &\quad d(G_1) - i_1 + i_2 - i, d(G_2) - i_2 + i_1 - i\}. \end{aligned}$$

Then G has a coloring φ such that $r(\varphi) = i$, $r(\varphi_1) = i_1$, and $r(\varphi_2) = i_2$ if and only if $0 \leq i \leq h_s(i_1, i_2)$. Furthermore any such coloring φ satisfies $\#\varphi \geq \alpha_s(i, i_1, i_2)$, and indeed G has such a coloring φ with $\#\varphi = \alpha_s(i, i_1, i_2)$.

Proof. Given in Appendix.

Lemma 3.4 immediately implies the following lemma.

LEMMA 3.5. *Let $G = G_1 \cdot G_2$, then the following hold:*

- (a) $i_{\min}(G) = 0$;
- (b) $i_{\max}(G) = \min\{\delta_{st}(G), d(G) - i_{\min}(G_1) - i_{\min}(G_2)\}$; and
- (c) *for every i , $i_{\min}(G) \leq i \leq i_{\max}(G)$,*

$$\chi'(G, i) = \min_{i_1, i_2} \alpha_s(i, i_1, i_2),$$

where the minimum is taken over all i_1 and i_2 such that

$$\begin{aligned} i_{\min}(G_1) &\leq i_1 \leq i_{\max}(G_1), \\ i_{\min}(G_2) &\leq i_2 \leq i_{\max}(G_2), \quad \text{and} \\ i &\leq h_s(i_1, i_2). \end{aligned}$$

By Lemmas 3.3 and 3.5 we obtain the following straightforward algorithm. Let T_b be a binary decomposition tree of a series-parallel multigraph G , and let u be a node in T_b . Then the following procedure $\text{Color}(G_u)$ computes $i_{\min}(G_u)$, $i_{\max}(G_u)$ and $\chi'(G_u, i)$ for all i , $i_{\min}(G_u) \leq i \leq i_{\max}(G_u)$. We first call procedure $\text{Color}(G_r)$ for the root r of T_b , and then compute the chromatic index $\chi'(G) = \min\{\chi'(G_r, i) | i_{\min}(G_r) \leq i \leq i_{\max}(G_r)\}$, where $G = G_r$.

- 1 *Procedure* $\text{Color}(G_u)$;
 $\{G_u$ is the subgraph of G corresponding to node u of $T_b\}$
- 2 **begin**
- 3 **if** u is a leaf node **then begin**

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4      let  $m$  be the number of multiple edges in  $G_u$ ;
5       $i_{\min}(G_u) := i_{\max}(G_u) := m$ ;
6       $\chi'(G_u, m) := m$ 
7  end
8  else if  $u$  is a p-node of  $T_b$  then begin
9      let  $v_1$  and  $v_2$  be the children of node  $u$  in  $T_b$ ;     $\{G_u = G_{v_1} \| G_{v_2}\}$ 
10     Color( $G_{v_1}$ ); Color( $G_{v_2}$ );
11     compute  $i_{\min}(G_u), i_{\max}(G_u)$  and all  $\chi'(G_u, i)$  by using
        Lemma 3.3
12     end
13     else begin
14         let  $v_1$  and  $v_2$  be the children of node  $u$  in  $T_b$ ;     $\{G_u = G_{v_1} \cdot G_{v_2}\}$ 
15         Color( $G_{v_1}$ ); Color( $G_{v_2}$ );
16         compute  $i_{\min}(G_u), i_{\max}(G_u)$  and all  $\chi'(G_u, i)$  by
            using Lemma 3.5
17         end
18 end;
```

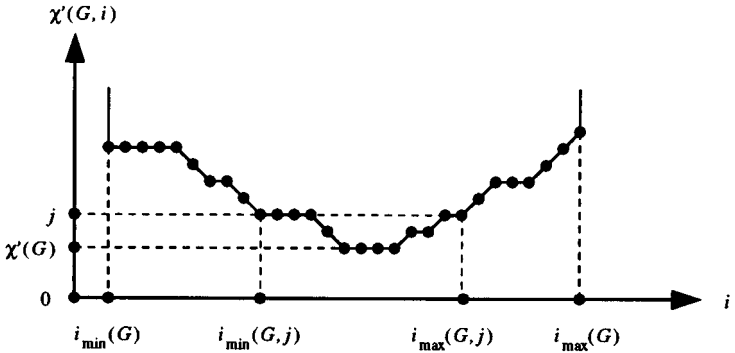
The variables i , i_1 , and i_2 range from 0 to at most Δ . Therefore Lines 11 and 16 in the algorithm above can be done in time $O(\Delta^3)$. If $G' = (V, E')$ is an underlying *simple* graph of the series-parallel *multigraph* $G = (V, E)$, then $|E'| \leq 2n - 3$. Therefore T_b contains $2n - 3$ leaves and hence $4n - 7$ nodes in total where $n = |V|$. Consequently the recursive calls occur at most $4n - 7$ times during the execution of Color(G_r). Thus the total running time of the algorithm is $O(n\Delta^3)$. An early version [13] of this paper improved the complexity to $O(n\Delta)$.

4. LINEAR ALGORITHM

In Section 3 we have shown that $\chi'(G, i)$ for all i and hence $\chi'(G)$ can be computed in $O(n\Delta^3)$ time. In this section we improve the time complexity to linear time; that is, we give an $O(|E|)$ time sequential algorithm to decide the chromatic index $\chi'(G)$ of a series-parallel multigraph $G = (V, E)$. Although we will not give the details, the algorithm can be modified so that it actually finds an edge-coloring of G using $\chi'(G)$ colors.

The first idea behind the linear algorithm is to observe that $\chi'(G, i)$ is a convex and “unit-staircase” function with respect to i as illustrated in Fig. 6. That is, the following two lemmas hold.

LEMMA 4.1. *Let G be a series-parallel multigraph. Then $|\chi'(G, i) - \chi'(G, i + 1)| \leq 1$ if $i_{\min}(G) \leq i < i_{\max}(G)$.*

FIG. 6. Illustration for functions $\chi'(G, i)$, $i_{\min}(G, j)$, and $i_{\max}(G, j)$.

Proof. By Lemma 3.1, $\chi'(G, i) \leq \chi'(G, i + 1) + 1$. Therefore it suffices to prove that $\chi'(G, i + 1) \leq \chi'(G, i) + 1$. Let φ be a coloring of G such that $\#\varphi = \chi'(G, i)$ and $r(\varphi) = i$. Since $\chi'(G, i + 1) \neq \infty$, $S(\varphi) \neq \emptyset$ and $T(\varphi) \neq \emptyset$. Suppose that there exist two nonadjacent edges (v_s, v_1) and (v_t, v_2) such that $\varphi((v_s, v_1)) \in S(\varphi)$, $\varphi((v_t, v_2)) \in T(\varphi)$, and $v_1 \neq v_2$. Then, recoloring these two edges with the same new color, one can construct a coloring φ' such that $\#\varphi' \leq \#\varphi + 1$ and $r(\varphi') = i + 1$, and hence $\chi'(G, i + 1) \leq \chi'(G, i) + 1$. Thus we may assume that all the edges whose colors are in $S(\varphi) \cup T(\varphi)$ and which are incident with v_s or v_t share a single common vertex, say v . Let $E_s = \{(v_s, u) \in E | u \neq v, v_t\}$ and $E_t = \{(v_t, w) \in E | w \neq v, v_s\}$. Then clearly $R(\varphi) = \varphi(E_s \cup E_t \cup E(v_s, v_t))$ and $\varphi(E_s \cup E_t) \cap \varphi(E(v_s, v_t)) = \emptyset$. On the other hand, any coloring μ of G satisfies $R(\mu) \subseteq \mu(E_s \cup E_t \cup E(v_s, v_t))$, and G has a coloring μ with $r(\mu) = i + 1$, and hence $i + 1 \leq |E_s \cup E_t \cup E(v_s, v_t)|$. Therefore $\varphi(E_s) \cap \varphi(E_t) \neq \emptyset$; otherwise, $\varphi(E_s)$, $\varphi(E_t)$, and $\varphi(E(v_s, v_t))$ are pairwise disjoint and hence $i = r(\varphi) = |E_s \cup E_t \cup E(v_s, v_t)|$, a contradiction. Hence there exists a pair of edges $e_1 = (v_s, x) \in E_s$ and $e_2 = (v_t, y) \in E_t$ such that $\varphi(e_1) = \varphi(e_2)$. Let $e_3 = (v_s, v)$ and $e_4 = (v_t, v)$ be a pair of edges such that $\varphi(e_3) \in S(\varphi)$ and $\varphi(e_4) \in T(\varphi)$. Let $c_k = \varphi(e_k)$, $k = 1, 2, 3$, and 4 . Note that $c_1 = c_2$. Let c_0 be a new color. We can construct from φ a coloring φ' of G such that $\#\varphi' \leq \#\varphi + 1$ and $r(\varphi') = i + 1$ as follows:

$$\varphi'(e) = \begin{cases} c_0 & \text{if } e = e_1, e_4; \\ c_4 & \text{if } e = e_2, e_3; \\ c_1 & \text{if } e = (x, y) \text{ and } \varphi(e) = c_4; \\ c_0 & \text{if } e = (z, y), z \neq x \text{ and } \varphi(e) = c_4; \\ \varphi(e) & \text{otherwise.} \end{cases} \quad \text{Q.E.D.}$$

LEMMA 4.2. *Let G be a series-parallel multigraph, and let l and h be integers. Then for any integer i such that $l \leq i \leq h$*

$$\chi'(G, i) \leq \max\{\chi'(G, l), \chi'(G, h)\}.$$

Proof. Suppose for a contradiction that the lemma does not hold for a series-parallel multigraph G and some integers l and h . We may assume that the number of edges in G is minimum among such graphs and that $h - l$ is minimum among such integers for G . Obviously $h - l \geq 2$. Furthermore, from Lemma 4.1 and the minimality of $h - l$, one can know that $\chi'(G, i) > \chi'(G, l) = \chi'(G, h) \neq \infty$ for all $i, l < i < h$. Let $j = \chi'(G, l) = \chi'(G, h)$. Clearly G has at least two edges, and hence either $G = G_1 \| G_2$ or $G = G_1 \cdot G_2$ for series-parallel multigraphs G_1 and G_2 . We give a proof only for the case $G = G_1 \| G_2$ since the proof for the case $G = G_1 \cdot G_2$ is similar.

Since $\chi'(G, h) = j$, by Lemma 3.3(c) there exist two integers i_1 and i_2 such that $i_1 + i_2 \leq h \leq h_p(i_1, i_2)$ and $j = \alpha_p(h, i_1, i_2) = \max\{\chi'(G_1, i_1), \chi'(G_2, i_2), d(G) - h\}$. Clearly

$$j = \chi'(G, l) \geq d(G) - l \geq d(G) - h + 2, \quad (1)$$

and hence $j = \max\{\chi'(G_1, i_1), \chi'(G_2, i_2)\}$. We now claim that

$$i_1 + i_2 = h. \quad (2)$$

Otherwise, $i_1 + i_2 \leq h - 1 \leq h_p(i_1, i_2)$ and hence by Lemma 3.3(c) and (1)

$$\begin{aligned} \chi'(G, h - 1) &\leq \alpha_p(h - 1, i_1, i_2) \\ &= \max\{\chi'(G_1, i_1), \chi'(G_2, i_2), d(G) - (h - 1)\} \\ &= j, \end{aligned}$$

contradicting the fact that $\chi'(G, i) > j$ for all $i, l < i < h$.

Since $\chi'(G, l) = j$, by Lemma 3.3(c) there exist two integers i'_1 and i'_2 such that $i'_1 + i'_2 \leq l \leq h_p(i'_1, i'_2)$ and $j = \alpha_p(l, i'_1, i'_2) = \max\{\chi'(G_1, i'_1), \chi'(G_2, i'_2), d(G) - l\}$. Since $i_1 + i_2 = h \geq l + 2 \geq i'_1 + i'_2 + 2$, either $i_1 > i'_1$ or $i_2 > i'_2$. Assume without loss of generality that $i_1 > i'_1$, and choose i_1^* such that $i_1^* = i_1 - 1$, then $i_1 > i_1^* \geq i'_1$. Clearly $\chi'(G_1, i_1) \leq j$ and $\chi'(G_1, i'_1) \leq j$. By the assumption the lemma holds for the subgraph G_1 of G . Therefore $\chi'(G_1, i_1^*) \leq \max\{\chi'(G_1, i_1), \chi'(G_1, i'_1)\} \leq j$. Since $i_1^* + i_2 = h - 1 \leq h_p(i_1, i_2) - 1 \leq h_p(i_1^*, i_2)$, by Lemma 3.3(c)

$$\begin{aligned} \chi'(G, h - 1) &\leq \alpha_p(h - 1, i_1^*, i_2) \\ &= \max\{\chi'(G_1, i_1^*), \chi'(G_2, i_2), d(G) - (h - 1)\}. \end{aligned}$$

Since $\chi'(G_1, i_1^*) \leq j$, $\chi'(G_2, i_2) \leq j$ and $d(G) - (h - 1) \leq j$, we have $\chi'(G, h - 1) \leq j$, contradicting the fact that $\chi'(G, i) > j$ for all $i, l < i < h$.
Q.E.D.

Since $\chi'(G, i)$ is a convex function, one can consider a kind of inverse functions of $\chi'(G, i)$, which are called $i_{\min}(G, j)$ and $i_{\max}(G, j)$ and defined for an integer j as follows:

$$i_{\min}(G, j) = \begin{cases} \min\{i \mid \chi'(G, i) \leq j\} & \text{if } j \geq \chi'(G), \\ +\infty & \text{otherwise} \end{cases}$$

and

$$i_{\max}(G, j) = \begin{cases} \max\{i \mid \chi'(G, i) \leq j\} & \text{if } j \geq \chi'(G), \\ -\infty & \text{otherwise.} \end{cases}$$

Then $i_{\min}(G) = \min\{i_{\min}(G, j) \mid j \geq 0\}$ and $i_{\max}(G) = \max\{i_{\max}(G, j) \mid j \geq 0\}$. Furthermore $\chi'(G, i) \leq j$ if and only if $i_{\min}(G, j) \leq i \leq i_{\max}(G, j)$. See Fig. 6.

The second idea behind the linear algorithm is to recursively compute $\chi'(G)$ by using $i_{\min}(G, j)$ and $i_{\max}(G, j)$ in place of $\chi'(G, i)$. By Lemma 4.1, if $\chi'(G, i) \neq \infty$ then $\chi'(G, i) \leq \chi'(G) + \delta_{\text{st}}(G)$. Therefore it suffices to compute $i_{\min}(G, j)$ and $i_{\max}(G, j)$ only for j such that $\chi'(G) \leq j \leq \chi'(G) + \delta_{\text{st}}(G)$. Define a *min-max triple set* $\mathcal{A}(G)$ as follows:

$$\mathcal{A}(G) = \{(j, i_{\min}(G, j), i_{\max}(G, j)) \mid \chi'(G) \leq j \leq \chi'(G) + \delta_{\text{st}}(G)\}.$$

Then one can compute $\chi'(G)$ from $\mathcal{A}(G)$ in $O(\delta_{\text{st}}(G))$ time since $\chi'(G) = \min\{j \mid (j, i_{\min}(G, j), i_{\max}(G, j)) \in \mathcal{A}(G)\}$.

The third idea is that $\mathcal{A}(G)$ can be computed directly from $\mathcal{A}(G_1)$ and $\mathcal{A}(G_2)$ when $G = G_1 \cdot G_2$ or $G_1 \parallel G_2$. When one computes $\mathcal{A}(G)$ from $\mathcal{A}(G_1)$ and $\mathcal{A}(G_2)$, one needs to know $\chi'(G)$, since j ranges from $\chi'(G)$ to $\chi'(G) + \delta_{\text{st}}(G)$. One can indeed compute $\mathcal{A}(G)$ and $\chi'(G)$ from $\mathcal{A}(G_1)$ and $\mathcal{A}(G_2)$ as shown in Lemmas 4.3 and 4.4.

LEMMA 4.3. *Let $G = G_1 \parallel G_2$ and $j \geq \chi'(G)$; then the following (a)–(c) hold:*

- (a) $i_{\min}(G, j) = \max\{i_{\min}(G_1, j) + i_{\min}(G_2, j), d(G) - j\};$
- (b) $i_{\max}(G, j) = \min\{\delta_{\text{st}}(G), d(G_1) - i_{\min}(G_1, j) + i_{\max}(G_2, j), d(G_2) - i_{\min}(G_2, j) + i_{\max}(G_1, j)\};$ and
- (c) $\chi'(G) = \min\{x \geq b_2 \mid x \geq d(G_1) - i_{\max}(G_1, x) + i_{\min}(G_2, x), x \geq d(G_2) - i_{\max}(G_2, x) + i_{\min}(G_1, x)\},$ where $b_2 = \max\{\Delta_{\text{st}}(G), \chi'(G_1), \chi'(G_2)\}$ is a trivial lower bound on $\chi'(G)$.

Proof. (a) Lemma 3.3(c) implies

$$i_{\min}(G, j) = \min_{i_1, i_2} \{i | i_1 + i_2 \leq i \leq h_p(i_1, i_2) \text{ and } \alpha_p(i, i_1, i_2) \leq j\},$$

where the minimum is taken over all integers i_1 and i_2 such that

$$i_{\min}(G_1, j) \leq i_1 \leq i_{\max}(G_1, j) \quad \text{and} \quad (3)$$

$$i_{\min}(G_2, j) \leq i_2 \leq i_{\max}(G_2, j). \quad (4)$$

Clearly $\alpha_p(i, i_1, i_2) \leq j$ if and only if the three equations $d(G) - j \leq i$, (3) and (4) hold. Therefore

$$i_{\min}(G, j) = \min_{i_1, i_2} \{i \geq d(G) - j | i_1 + i_2 \leq i \leq h_p(i_1, i_2)\}, \quad (5)$$

where i_1 and i_2 run over the same ranges above. It suffices to consider the following three cases.

Case 1. $d(G) - j < i_{\min}(G_1, j) + i_{\min}(G_2, j)$. In this case by (5) $i_{\min}(G, j) = i_{\min}(G_1, j) + i_{\min}(G_2, j)$, and hence (a) holds.

Case 2. $i_{\min}(G_1, j) + i_{\min}(G_2, j) \leq d(G) - j \leq i_{\max}(G_1, j) + i_{\max}(G_2, j)$. In this case there are two integers i_1 and i_2 satisfying (3), (4) and $i_1 + i_2 = d(G) - j$. Therefore by (5) $i_{\min}(G, j) = d(G) - j$, and hence (a) holds.

Case 3. $i_{\max}(G_1, j) + i_{\max}(G_2, j) < d(G) - j$. Since $j \geq \chi'(G)$, by Lemma 3.3(c) there exist integers i, i_1 , and i_2 satisfying the four equations (3), (4), $\alpha_p(i, i_1, i_2) \leq j$, and $i_1 + i_2 \leq i \leq h_p(i_1, i_2)$. Since $\alpha_p(i, i_1, i_2) \leq j$, $d(G) - j \leq i$. Furthermore

$$i_1 + i_2 \leq i_{\max}(G_1, j) + i_{\max}(G_2, j) < d(G) - j \leq i \leq h_p(i_1, i_2).$$

Therefore by (5) $i_{\min}(G, j) = d(G) - j$, and hence (a) holds.

(b) is similar to (a) above.

(c) By Lemma 3.3(c) we have

$$\chi'(G) = \min_{i, i_1, i_2} \{\alpha_p(i, i_1, i_2) | i_1 + i_2 \leq i \leq h_p(i_1, i_2)\}.$$

Since $\alpha_p(i, i_1, i_2)$ is non-increasing with respect to i for fixed i_1 and i_2 , the minimum above is attained when $i = h_p(i_1, i_2)$. Therefore

$$\chi'(G) = \min_{i_1, i_2} \alpha_p(h_p(i_1, i_2), i_1, i_2) = \min_{i_1, i_2} \{x | x \geq \alpha_p(h_p(i_1, i_2), i_1, i_2)\}.$$

By the definition of α_p we have

$$\chi'(G) = \min_{i_1, i_2} \{x | x \geq \chi'(G_1, i_1), x \geq \chi'(G_2, i_2), x \geq d(G) - h_p(i_1, i_2)\}.$$

The minimum above is taken over all integers i_1 and i_2 such that $i_{\min}(G_1) \leq i_1 \leq i_{\max}(G_1)$ and $i_{\min}(G_2) \leq i_2 \leq i_{\max}(G_2)$. Obviously $\chi'(G_1, i_1) \leq x \neq \infty$ if and only if $\chi'(G_1) \leq x$ and $i_{\min}(G_1, x) \leq i_1 \leq i_{\max}(G_1, x)$. Similarly $\chi'(G_2, i_2) \leq x \neq \infty$ if and only if $\chi'(G_2) \leq x$ and $i_{\min}(G_2, x) \leq i_2 \leq i_{\max}(G_2, x)$. Thus we have

$$\chi'(G) = \min_{i_1, i_2} \{x | x \geq \chi'(G_1), x \geq \chi'(G_2), x \geq d(G) - h_p(i_1, i_2)\}$$

where the minimum is taken over all integers i_1 and i_2 such that $i_{\min}(G_1, x) \leq i_1 \leq i_{\max}(G_1, x)$ and $i_{\min}(G_2, x) \leq i_2 \leq i_{\max}(G_2, x)$. Therefore by the definition of h_p and $d(G) - \delta_{st}(G) = \Delta_{st}(G) \geq d(G)/2$, we have

$$\begin{aligned} \chi'(G) = \min \{ & x \geq b_2 | x \geq d(G_1) - i_{\max}(G_1, x) + i_{\min}(G_2, x), \\ & x \geq d(G_2) - i_{\max}(G_2, x) + i_{\min}(G_1, x) \}. \end{aligned}$$

Q.E.D

For the case $G = G_1 \cdot G_2$ we have Lemma 4.4

LEMMA 4.4. *Let $G = G_1 \cdot G_2$, $v = v_t(G_1) = v_s(G_2)$, and $j \geq \chi'(G)$, then the following (a)–(c) hold:*

- (a) $i_{\min}(G, j) = \max\{j, d(G), d(G_1) - i_{\max}(G_1, j) + i_{\min}(G_2, j), d(G_2) - i_{\max}(G_2, j) + i_{\min}(G_1, j)\} - j$;
- (b) $i_{\max}(G, j) = \min\{\delta_{st}(G), d(G) - i_{\min}(G_1, j) - i_{\min}(G_2, j), j - d(v) + i_{\max}(G_1, j) + i_{\max}(G_2, j)\}$; and
- (c) $\chi'(G) = \max\{\chi'(G_1), \chi'(G_2), d(v)\}$.

Proof. (a) Lemma 3.5(c) implies

$$i_{\min}(G, j) = \min_{i_1, i_2} \{i | 0 \leq i \leq h_s(i_1, i_2) \text{ and } \alpha_s(i, i_1, i_2) \leq j\},$$

where the minimum is taken over all integers i_1 and i_2 such that

$$i_{\min}(G_1, j) \leq i_1 \leq i_{\max}(G_1, j) \quad \text{and} \quad (6)$$

$$i_{\min}(G_2, j) \leq i_2 \leq i_{\max}(G_2, j). \quad (7)$$

The definition of α_s implies that $\alpha_s(i, i_1, i_2) \leq j$ if and only if the four equations (6), (7), $d(v) \leq j$, and

$$\begin{aligned} \max\{ & d(G) - i, d(v) - i_1 - i_2 + i, d(G_1) - i_1 + i_2 - i, d(G_2) \\ & - i_2 + i_1 - i \} \leq j \end{aligned}$$

hold. Therefore

$$i_{\min}(G, j) = \min_{i_1, i_2} \{i \geq 0 | i \leq h_s(i_1, i_2), i \leq j - d(v) + i_1 + i_2, i \geq d(G) - j, \\ i \geq d(G_1) - i_1 + i_2 - j, i \geq d(G_2) - i_2 + i_1 - j\}.$$

We now claim that

$$h_s(i_1, i_2) \geq \max\{d(G) - j, d(G_1) - i_1 + i_2 - j, d(G_2) - i_2 + i_1 - j\}. \quad (8)$$

We first prove

$$h_s(i_1, i_2) \geq d(G) - j, \quad (9)$$

that is,

$$j \geq d(G) - h_s(i_1, i_2) = \max\{d(G) - \delta_{\text{st}}(G), i_1 + i_2\}. \quad (10)$$

Note that $h_s(i_1, i_2) = \min\{\delta_{\text{st}}(G), d(G) - i_1 - i_2\}$. Clearly $j \geq \chi'(G) \geq \Delta_{\text{st}}(G) = d(G) - \delta_{\text{st}}(G)$, and $j \geq d(v) \geq i_1 + i_2$. Thus we have verified (10) and hence (9). We next prove

$$h_s(i_1, i_2) \geq d(G_1) - i_1 + i_2 - j, \quad (11)$$

that is,

$$j \geq \max\{d(G_1) - i_1 + i_2 - \delta_{\text{st}}(G), d_1(v) - d(v_t) + 2i_2\}. \quad (12)$$

Since $\delta_{\text{st}}(G) = \min\{d(v_s), d(v_t)\}$ and $d(G_1) = d(v_s) + d_1(v)$, we shall prove that

$$j \geq \max\{d_1(v) - i_1 + i_2, d(G_1) - i_1 + i_2 - d(v_t), d_1(v) - d(v_t) + 2i_2\}. \quad (13)$$

Clearly $j \geq d(v) \geq d_1(v) + i_2$. Since $j \geq \chi'(G_1, i_1) \geq d(G_1) - i_1$ and $i_2 - d(v_t) \leq 0$, we have $j \geq d(G_1) - i_1 + i_2 - d(v_t)$. Similarly we have $j \geq (d_1(v) + i_2) + (i_2 - d(v_t)) = d_1(v) - d(v_t) + 2i_2$. Thus we have verified (13) and hence (11). As in the proof of (11), one can prove that

$$h_s(i_1, i_2) \geq d(G_2) - i_2 + i_1 - j. \quad (14)$$

The three equations (9), (11), and (14) immediately imply (8).

As above, one can easily verify that

$$j - d(v) + i_1 + i_2 \\ \geq \max\{d(G) - j, d(G_1) - i_1 + i_2 - j, d(G_2) - i_2 + i_1 - j\}. \quad (15)$$

Obviously the following equation holds for any $c_1, c_2, x_{\min}$, and $x_{\max}$ such that $x_{\min} \leq x_{\max}$:

$$\begin{aligned} & \min_{x_{\min} \leq x \leq x_{\max}} \{i | i \geq c_1 - x, i \geq c_2 + x\} \\ &= \max \left\{ \frac{c_1 + c_2}{2}, c_1 - x_{\max}, c_2 + x_{\min} \right\}. \end{aligned} \quad (16)$$

By (8), (15), (16), and $j \geq \chi'(G) \geq d(v)$ we have

$$\begin{aligned} i_{\min}(G, j) &= \min_{i_1, i_2} \{i \geq 0 | i \geq d(G) - j, \\ &\quad i \geq d(G_1) - i_1 + i_2 - j, i \geq d(G_2) - i_2 + i_1 - j\} \\ &= \max \left\{ 0, d(G) - j, \frac{d(G_1) + d(G_2)}{2} - j, \right. \\ &\quad d(G_1) - i_{\max}(G_1, j) + i_{\min}(G_2, j) - j, \\ &\quad \left. d(G_2) - i_{\max}(G_2, j) + i_{\min}(G_1, j) - j \right\} \\ &= \max \left\{ 0, d(G) - j, \frac{d(G) + d(v)}{2} - j, \right. \\ &\quad d(G_1) - i_{\max}(G_1, j) + i_{\min}(G_2, j) - j, \\ &\quad \left. d(G_2) - i_{\max}(G_2, j) + i_{\min}(G_1, j) - j \right\} \\ &= \max \left\{ 0, d(G) - j, \frac{d(G) - j}{2} + \frac{d(v) - j}{2}, \right. \\ &\quad d(G_1) - i_{\max}(G_1, j) + i_{\min}(G_2, j) - j, \\ &\quad \left. d(G_2) - i_{\max}(G_2, j) + i_{\min}(G_1, j) - j \right\} \\ &= \max \{ 0, d(G) - j, d(G_1) - i_{\max}(G_1, j) + i_{\min}(G_2, j) - j, \\ &\quad d(G_2) - i_{\max}(G_2, j) + i_{\min}(G_1, j) - j \} \\ &= \max \{ j, d(G), d(G_1) - i_{\max}(G_1, j) + i_{\min}(G_2, j), \\ &\quad d(G_2) - i_{\max}(G_2, j) + i_{\min}(G_1, j) \} - j. \end{aligned}$$

(b) is similar to (a) above.

(c) Trivially $\chi'(G) \geq \max\{\chi'(G_1), \chi'(G_2), d(v)\}$. Therefore we shall prove that $\chi'(G) \leq \max\{\chi'(G_1), \chi'(G_2), d(v)\}$. Let η and ψ be colorings of G_1 and G_2 , respectively, such that $\#\eta = \chi(G_1)$, $\#\psi = \chi'(G_2)$ and

$\eta(E(G_1)) \cap \psi(E(G_2)) = \phi$. Let φ be a coloring of G extended from η and ψ . Then $\#\varphi = \chi'(G_1) + \chi'(G_2)$. Therefore we shall decrease $\#\varphi$ to $\max\{\chi'(G_1), \chi'(G_2), d(v)\}$. One can assume w.l.o.g. that $\chi'(G_1) \geq \chi'(G_2)$. Therefore it suffices to consider the following two cases.

Case 1. $\chi'(G_1) \geq d(v)$.

Clearly $\chi'(G_1) = s(\eta) + t(\eta) + r(\eta) + q(\eta)$ and $d(v) = t(\eta) + r(\eta) + s(\psi) + r(\psi)$. Therefore in this case $s(\eta) + q(\eta) \geq s(\psi) + r(\psi)$. Since $\chi'(G_1) \geq \chi'(G_2)$, $\#\eta - (s(\psi) + r(\psi)) \geq t(\psi) + q(\psi)$. By the two equations above one can alter ψ so that ψ uses only colors in $\eta(E(G_1))$ as follows:

(i) replace the $s(\psi) + r(\psi)$ colors in $S(\psi) \cup R(\psi)$ with any $s(\psi) + r(\psi)$ colors in $S(\eta) \cup Q(\eta)$, and

(ii) replace the $t(\psi) + q(\psi)$ colors in $T(\psi) \cup Q(\psi)$ with any $t(\psi) + q(\psi)$ colors among the remaining $\#\eta - (s(\psi) + r(\psi))$ colors in $\eta(E(G_1))$.

The extension φ of η and this altered ψ is a coloring of G using $\#\eta$ colors. Hence $\chi'(G) \leq \#\eta = \chi'(G_1)$.

Case 2. $\chi'(G_1) < d(v)$.

Since $d(v) > \chi'(G_1) \geq \chi(G_2)$, we have

$$t(\eta) + r(\eta) > t(\psi) + q(\psi) \quad \text{and}$$

$$s(\psi) + r(\psi) > s(\eta) + q(\eta).$$

By the equations above one can alter η and ψ so that their extension φ uses only the $d(v)$ colors in $T(\eta) \cup R(\eta) \cup S(\psi) \cup R(\psi)$ as follows:

(i) replace the $t(\psi) + q(\psi)$ colors in $T(\psi) \cup Q(\psi)$ with any $t(\psi) + q(\psi)$ colors in $T(\eta) \cup R(\eta)$, and

(ii) replace the $s(\eta) + q(\eta)$ colors in $S(\eta) \cup Q(\eta)$ with any $s(\eta) + q(\eta)$ colors in $S(\psi) \cup R(\psi)$.

The extension φ of these altered η and ψ is a coloring of G using $d(v)$ colors. Hence $\chi'(G) \leq d(v)$. Q.E.D.

From these lemmas we have the following lemma. In the analysis of time-complexity we assume that the degrees of terminals of a series-parallel multigraph G and its components G_1 and G_2 are known, because the bookkeeping operations to compute them take linear time in total.

LEMMA 4.5. *If $G = G_1 \cdot G_2$ or $G = G_1 \parallel G_2$, then one can compute $\mathcal{A}(G)$ from $\mathcal{A}(G_1)$ and $\mathcal{A}(G_2)$ in time $O(\delta_{\text{st}}(G))$.*

Proof. Lemmas 4.3 and 4.4 imply that $i_{\min}(G, j)$ and $i_{\max}(G, j)$ can be computed in $O(1)$ time for a fixed integer j . Therefore $i_{\min}(G, j)$ and $i_{\max}(G, j)$ for all j , $\chi'(G) \leq j \leq \chi'(G) + \delta_{\text{st}}(G)$, can be computed in

$O(\delta_{\text{st}}(G))$ time. Hence $\mathcal{A}(G)$ can be computed in $O(\delta_{\text{st}}(G))$ time if $\chi'(G)$ is known. Thus we shall show that $\chi'(G)$ can be computed in $O(\delta_{\text{st}}(G))$ time.

If $G = G_1 \cdot G_2$, then by Lemma 4.4(c) $\chi'(G)$ can be computed in $O(1)$ time. If $G = G_1 \| G_2$, then by Lemma 4.3(c) one can compute $\chi'(G)$ in $O(\delta_{\text{st}}(G))$ time because it suffices to take the minimum only for x such that $b_2 \leq x \leq b_2 + \max\{\delta_{\text{st}}(G_1), \delta_{\text{st}}(G_2)\}$. Note that by Lemma 4.1 $i_{\min}(G_i, x)$ and $i_{\max}(G_i, x)$, $i = 1, 2$, are constants for all $x \geq \chi'(G_i) + \delta_{\text{st}}(G_i)$; that is, $i_{\min}(G_i, x) = i_{\min}(G_i)$ and $i_{\max}(G_i, x) = i_{\max}(G_i)$. (See Fig. 6.) Q.E.D.

The three ideas above are not sufficient to provide a linear algorithm. The fourth idea is to process all the components connected in parallel together. That is, when $G = G_1 \| G_2 \| \cdots \| G_k$, we do not compute $\mathcal{A}(G_1), \mathcal{A}(G_1 \| G_2), \dots, \mathcal{A}(G_1 \| G_2 \| \cdots \| G_k)$ sequentially in this order, but compute $\mathcal{A}(G)$ directly from $\mathcal{A}(G_1), \mathcal{A}(G_2), \dots, \mathcal{A}(G_k)$. Note that $\delta_{\text{st}}(G_1) + \delta_{\text{st}}(G_1 \| G_2) + \cdots + \delta_{\text{st}}(G_1 \| G_2 \| \cdots \| G_k)$ may become $O(n^2)$. Thus we do not consider an ordinary binary decomposition tree T_b of a series-parallel multigraph G , but consider a “*reduced (non-binary) decomposition tree*” T_{red} in which there is no edge joining two p-nodes. T_{red} can be obtained from T_b by contracting every edge joining two p-nodes. Figure 7 illustrates such a reduced decomposition tree T_{red} of a series-parallel multigraph in Fig. 2a. We have the following Lemma 4.6. Since it is rather straightforward to prove Lemma 4.6 by induction on k using Lemma 4.3, the proof is omitted in this paper.

LEMMA 4.6. *If $G = G_1 \| G_2 \| \cdots \| G_k$ and $j \geq \chi'(G)$, then*

$$i_{\min}(G, j) = \max\{\text{sum}(j), d(G) - j\},$$

$$i_{\max}(G, j) = \min\{\delta_{\text{st}}(G), d(G) - \text{sum}(j) + m(j)\}, \text{ and}$$

$$\chi'(G) = \min\{x \geq b_1 \mid x \geq \text{sum}(x) - m(x)\},$$

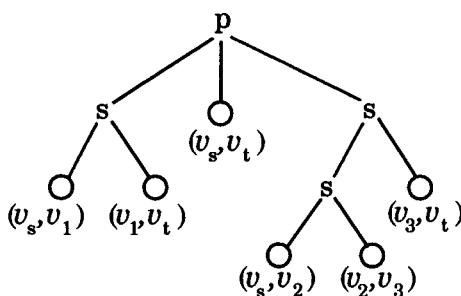


FIG. 7. Reduced decomposition tree T_{red} .

where

$$\text{sum}(x) = \sum_{p=1}^k i_{\min}(G_p, x),$$

$$m(x) = \min_{1 \leq p \leq k} \{i_{\min}(G_p, x) + i_{\max}(G_p, x) - d(G_p)\}, \quad \text{and}$$

$$b_1 = \max\{\Delta_{\text{st}}(G), \chi'(G_1), \chi'(G_2), \dots, \chi'(G_k)\}.$$

From this lemma we have the following lemma.

LEMMA 4.7. *If $G = G_1 \| G_2 \| \dots \| G_k$, then one can compute $\mathcal{A}(G)$ from $\mathcal{A}(G_1), \mathcal{A}(G_2), \dots, \mathcal{A}(G_k)$ in $O(\delta_{\text{st}}(G))$ time.*

Proof. By Lemma 4.6 one can compute $i_{\min}(G, j)$ and $i_{\max}(G, j)$ for a fixed j in time $O(k)$. Therefore a straightforward method takes $O(k\delta_{\text{st}}(G))$ time to compute $\mathcal{A}(G)$. However, we can compute $\mathcal{A}(G)$ in time $O(\delta_{\text{st}}(G))$ as follows.

Trivially, $b_1 \leq \chi'(G)$. Clearly $i_{\min}(G_p, x)$ and $i_{\max}(G_p, x)$ are constants for all $x \geq \chi'(G_p) + \delta_{\text{st}}(G_p)$; that is, $i_{\min}(G_p, x) = i_{\min}(G_p)$ and $i_{\max}(G_p, x) = i_{\max}(G_p)$. Let $b_h = \max_{1 \leq p \leq k} \{\chi'(G_p) + \delta_{\text{st}}(G_p)\}$; then $i_{\min}(G_p, x)$, $i_{\max}(G_p, x)$, and hence $\text{sum}(x), m(x)$ are all constants for all $x \geq b_h$. Therefore one needs to compute $\text{sum}(x)$ and $m(x)$ only for $x, b_1 \leq x \leq b_h$. The following algorithm computes $\text{sum}(x)$ for all $x, b_1 \leq x \leq b_h$, in time total $O(\delta_{\text{st}}(G))$. Note that $b_h - b_1 \leq \delta_{\text{st}}(G)$.

begin

$\text{sum}(b_1) := \sum_{p=1}^k i_{\min}(G_p, b_1);$

$K := \{p | 1 \leq p \leq k, b_1 < \chi'(G_p) + \delta_{\text{st}}(G_p)\};$

{if $p \in K$ then $i_{\min}(G_p, x)$ and $i_{\max}(G_p, x)$
may not be constants for current $x = b_1$ }

for $x := b_1 + 1$ **to** b_h **do**

begin

$\text{sum}(x) := \text{sum}(x - 1);$

for each $p \in K$ **do**

begin

$\text{sum}(x) := \text{sum}(x) - i_{\min}(G_p, x - 1) + i_{\min}(G_p, x);$

{update $\text{sum}(x)$ }

if $x \geq \chi'(G_p) + \delta_{\text{st}}(G_p)$ **then** $K := K - \{p\}$

end

end

end;

Since each $p, 1 \leq p \leq k$, is contained in the set K at most $\delta_{\text{st}}(G_p)$ times during the execution of the first **for**-loop, the algorithm takes time

$O(\sum_{p=1}^k \delta_{st}(G_p)) \leq O(\delta_{st}(G))$. Similarly one can compute all $m(x)$, $b_1 \leq x \leq b_h$, total in time $O(\delta_{st}(G))$. Using these values $\text{sum}(x)$ and $m(x)$ and three equations in Lemma 4.6, one can compute $\mathcal{A}(G)$ in $O(\delta_{st}(G))$ time. Q.E.D.

From the lemmas above we can obtain the following algorithm and theorem. One can compute the chromatic index $\chi'(G)$ of G by calling procedure $\text{Chromatic-Index}(G_r)$ for the root r of the reduced decomposition tree T_{red} .

Procedure Chromatic-Index(G_u);

begin

if u is a leaf node **then** $\mathcal{A}(G_u) := \{(m, m, m)\}$ where m is the number of multiple edges in G_u

else if u is a p-node of T_{red} **then begin**

let $v_1, v_2, \dots, v_k$ be the children of node u ;

$\{G_u = G_{v_1} \parallel \dots \parallel G_{v_k}\}$

for each i , $1 \leq i \leq k$, **do** Chromatic-Index(G_{v_i});

Compute $\mathcal{A}(G_u)$ from $\mathcal{A}(G_{v_1}), \mathcal{A}(G_{v_2}), \dots, \mathcal{A}(G_{v_k})$

end

else begin

let v_1 and v_2 be the children of node u ; $\{G_u = G_{v_1} \cdot G_{v_2}\}$

Chromatic-Index(G_{v_1}); Chromatic-Index(G_{v_2});

Compute $\mathcal{A}(G_u)$ from $\mathcal{A}(G_{v_1})$ and $\mathcal{A}(G_{v_2})$

end;

$\chi'(G_u) := \min\{j \mid (j, i_{\min}(G_u, j), i_{\max}(G_u, j)) \in \mathcal{A}(G_u)\}$

end;

THEOREM 4.8. *The chromatic index $\chi'(G)$ of a series-parallel multigraph $G = (V, E)$ can be decided on $O(|E|)$ time.*

Proof. The correctness of the algorithm above is an immediate consequence of Lemmas 4.4 and 4.6 above. Therefore we prove only the claim on the time complexity. Obviously one can compute $\chi'(G)$ from $\mathcal{A}(G)$ in $O(\delta_{st}(G))$ time. So it suffices to prove that $\mathcal{A}(G)$ can be computed in $O(|E(G)|)$ time. If $|V(G)| = 2$ then clearly one can compute $\mathcal{A}(G)$ in $O(|E|)$ time, say in $c|E|$ time units. By Lemma 4.7 there is a constant c_p such that $\mathcal{A}(G)$ can be computed from $\mathcal{A}(G_1), \mathcal{A}(G_2), \dots, \mathcal{A}(G_k)$ in $c_p \delta_{st}(G)$ time units when $G = G_1 \parallel G_2 \parallel \dots \parallel G_k$. Similarly, by Lemma 4.5 there is a constant c_s such that $\mathcal{A}(G)$ can be computed from $\mathcal{A}(G_1)$ and $\mathcal{A}(G_2)$ in $c_s \delta_{st}(G)$ time units when $G = G_1 \cdot G_2$. Let $c' = \max\{c_p, c_s\}$ and $c'' = c + 3c'$. Denote by $T_p(G)$ the running time of the algorithm to compute $\mathcal{A}(G)$ when either $|V(G)| = 2$ or $G = G_1 \parallel G_2 \parallel \dots \parallel G_k$, and denote by $T_s(G)$ the running time of the algorithm to compute $\mathcal{A}(G)$ when either $|V(G)| = 2$ or

$G = G_1 \cdot G_2$. We claim that the following two equations hold:

$$T_p(G) \leq c'' |E(G)| - 2c' \Delta_{\text{st}}(G), \quad \text{and} \quad (17)$$

$$T_s(G) \leq c'' |E(G)| - 2c' \Delta_{\text{st}}(G) - c' \delta_{\text{st}}(G). \quad (18)$$

We prove (17) and (18) by induction on the number of vertices in G . Clearly (17) and (18) hold when $|V(G)| = 2$. Assuming that Eq. (17) and (18) hold for $G_1, G_2, \dots, G_k$, we prove (17) and (18) for G . If $G = G_1 \| G_2 \| \dots \| G_k$, then

$$\begin{aligned} T_p(G) &\leq T_s(G_1) + T_s(G_2) + \dots + T_s(G_k) + c' \delta_{\text{st}}(G) \\ &\leq c'' |E(G)| - 2c' \Delta_{\text{st}}(G) - c' \delta_{\text{st}}(G) + c' \delta_{\text{st}}(G) \\ &= c'' |E(G)| - 2c' \Delta_{\text{st}}(G). \end{aligned}$$

If $G = G_1 \cdot G_2$, then

$$\begin{aligned} T_s(G) &\leq \max\{T_p(G_1), T_s(G_1)\} + \max\{T_p(G_2), T_s(G_2)\} + c' \delta_{\text{st}}(G) \\ &\leq c'' |E(G)| - 2c' \Delta_{\text{st}}(G_1) - 2c' \Delta_{\text{st}}(G_2) + c' \delta_{\text{st}}(G) \\ &\leq c'' |E(G)| - 2c' d(v_s) - 2c' d(v_t) + c' \delta_{\text{st}}(G) \\ &= c'' |E(G)| - 2c' \Delta_{\text{st}}(G) - c' \delta_{\text{st}}(G). \end{aligned} \quad \text{Q.E.D.}$$

Remark. The algorithm Chromatic-Index above decides the chromatic index $\chi'(G)$ of a given series-parallel multigraph G . One can derive, from the algorithm, an algorithm which actually finds an edge-coloring of G using $\chi'(G)$ colors. The edge-coloring algorithm runs in three passes as follows. In the first pass the algorithm decides $\chi'(G)$ by traversing the reduced decomposition tree T_{red} of the series-parallel multigraphs G from leaves to the root. In the second pass, traversing T_{red} from the root to leaves, the algorithm decides an appropriate integer i_p for every node n_p of T_{red} so that an edge-coloring φ_p , with $r(\varphi_p) = i_p$ and $\#\varphi_p = \chi'(G_p, i_p)$, of the subgraph G_p of G corresponding to node n_p can be constructed from edge-colorings $\varphi_{p_1}, \varphi_{p_2}, \dots, \varphi_{p_k}$ of the subgraphs corresponding to the children $n_{p_1}, n_{p_2}, \dots, n_{p_k}$ of n_p such that $r(\varphi_{p_l}) = i_{p_l}$ and $\#\varphi_{p_l} = \chi'(G_{p_l}, i_{p_l})$ for $1 \leq l \leq k$. Note that the integer i_p must be $i_{\min}(G, \chi'(G))$ when n_p is the root of T_{red} and that each node n_p preserves a min-max triple set $\mathcal{A}(G_p)$. In the third pass, traversing T_{red} from leaves to the root with modifying edge-colorings of subgraphs of G as described in the proofs of Lemmas 3.2 and 3.4, the algorithm actually constructs an edge-coloring φ of G such that $r(\varphi) = i_{\min}(G, \chi'(G))$ and $\#\varphi = \chi'(G)$. Clearly the first and second passes can be done total in linear time. On the other hand, the

third pass can be done total in time $O(|E|\log|E|)$. Note that one can construct an edge-coloring of a graph G_p from those of the two subgraphs G_1 and G_2 by modifying the edge-coloring of a smaller one in time $O(\min\{|E_1|, |E_2|\})$ where $G_p = G_1 \cdot G_2$ or $G_1 \| G_2$, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Thus such an edge-coloring algorithm runs in time $O(|E|\log|E|)$. Furthermore one can implement the edge-coloring algorithm so that it takes time $O(\Delta|V|)$ [13].

5. CONCLUSION

In this paper we have given a linear and practical algorithm for the edge-coloring problem on series-parallel multigraphs $G = (V, E)$. The complexity is optimal within a constant factor, and the constant is small. This algorithm can be implemented as an NC parallel algorithm, which takes $O(\log n)$ parallel time with $O(n \Delta / \log n)$ processors where $n = |V|$ [14].

Our algorithm solves a single particular problem, that is, the edge-coloring problem. However, the methods which we developed in this paper appear to be useful for many other problems, especially for the “edge-partition problem with respect to property π ” which asks that the edge set of a given graph be partitioned into a minimum number of subsets such that the subgraph induced by each subset satisfies the property π . For the edge-coloring problem, π is indeed a matching.

Zhou, Nakano, and Nishizeki recently obtained a linear algorithm for the edge-coloring problem on partial k -trees with bounded k [11, 12]. However, the algorithm does not work for series-parallel multigraphs. Note that a partial 2-tree has no multiple edges and hence a series-parallel multigraph is not always a partial 2-tree. The complexity of the algorithm is optimal within a constant factor, but the constant $2^{12k(k+1)}$ is huge even for $k = 2$.

APPENDIX. PROOF OF LEMMA 3.4

(a) *Necessity.* Let φ be a coloring of G such that $r(\varphi) = i$, $r(\varphi_1) = i_1$, and $r(\varphi_2) = i_2$. Clearly $R(\varphi) = \varphi(v_s) \cap \varphi(v_t)$, $|\varphi(v_s)| = d(v_s)$, and $|\varphi(v_t)| = d(v_t)$. Therefore $i \leq \min\{d(v_s), d(v_t)\} = \delta_{\text{st}}(G)$. All the i_1 colors in $R(\varphi_1)$ ($\subseteq \varphi(v_s)$) are different from all the i_2 colors in $R(\varphi_2)$ ($\subseteq \varphi(v_t)$) since they appear at vertex v . Therefore $R(\varphi) \subseteq (\varphi(v_s) - R(\varphi_1)) \cup (\varphi(v_t) - R(\varphi_2))$, and hence $i \leq d(v_s) - i_1 + d(v_t) - i_2 = d(G) - i_1 - i_2$. Thus we have proved that $0 \leq i \leq \min\{\delta_{\text{st}}(G), d(G) - i_1 - i_2\} = h_s(i_1, i_2)$.

We now prove that $\#\varphi \geq \alpha_s(i, i_1, i_2)$. Clearly $\#\varphi$ is not less than the first four terms of α_s : $\#\varphi \geq \chi'(G_1, i_1), \chi'(G_2, i_2), d(v), d(G) - i$. We then

show that $\#\varphi$ is not less than the fifth term $d(v) - i_1 - i_2 + i$. Clearly $\varphi(E) \supseteq T(\varphi_1) \cup S(\varphi_2) \cup R(\varphi)$, and the three sets $T(\varphi_1)$, $S(\varphi_2)$, and $R(\varphi)$ are pairwise disjoint. Therefore $\#\varphi \geq t(\varphi_1) + s(\varphi_2) + i$. Obviously $d(v) = d_1(v) + d_2(v) = (t(\varphi_1) + i_1) + (s(\varphi_2) + i_2)$. These two equations immediately imply that

$$\#\varphi \geq d(v) - i_1 - i_2 + i.$$

We next prove that $\#\varphi$ is not less than the sixth term $d(G_1) - i_1 + i_2 - i$. Clearly $\varphi(E) \supseteq S(\varphi_1) \cup T(\varphi_1) \cup R(\varphi_1) \cup (R(\varphi_2) - R(\varphi))$, and the four sets $S(\varphi_1)$, $T(\varphi_1)$, $R(\varphi_1)$ and $(R(\varphi_2) - R(\varphi))$ are pairwise disjoint. Therefore $\#\varphi \geq s(\varphi_1) + t(\varphi_1) + i_1 + (i_2 - i)$. Obviously $d(G_1) = s(\varphi_1) + t(\varphi_1) + 2i_1$. These two equations immediately imply

$$\#\varphi \geq d(G_1) - i_1 + i_2 - i.$$

Similarly one can prove that $\#\varphi$ is not less than the seventh term $d(G_2) - i_2 + i_1 - i$. Thus we have proved that $\#\varphi \geq \alpha_s(i, i_1, i_2)$.

(b) *Sufficiency.* Let i be an arbitrary integer such that $0 \leq i \leq h_s(i_1, i_2)$. We shall construct a coloring φ of G such that $r(\varphi) = i$, $r(\varphi_1) = i_1$, and $r(\varphi_2) = i_2$. The set $R(\varphi)$ for any coloring φ can be partitioned into three subsets as follows (see Fig. 8):

$$R(\varphi) = A \cup B \cup C,$$

$$A = S(\varphi_1) \cap R(\varphi_2),$$

$$B = R(\varphi_1) \cap T(\varphi_2), \quad \text{and}$$

$$C = S(\varphi_1) \cap T(\varphi_2).$$

Then $r(\varphi) = a + b + c$ where $a = |A|$, $b = |B|$, and $c = |C|$. By Lemma 3.1(b) G_1 has a coloring η such that $\#\eta = \chi'(G_1, i_1)$ and $r(\eta) = i_1$. Similarly, G_2 has a coloring ψ such that $\#\psi = \chi'(G_2, i_2)$ and $r(\psi) = i_2$. One may assume w.l.o.g. that $\eta(E(G_1)) \cap \psi(E(G_2)) = \emptyset$. Consider a coloring φ of G which is an extension of η and ψ . Then $r(\varphi_1) = i_1$ and $r(\varphi_2) = i_2$, but $r(\varphi) = 0$ and $\#\varphi = \chi'(G_1, i_1) + \chi'(G_2, i_2)$. Therefore we shall alter the coloring φ to increase $r(\varphi)$ from 0 to i and decrease $\#\varphi$ from $\chi'(G_1, i_1) + \chi'(G_2, i_2)$ to $\alpha_s(i, i_1, i_2)$.

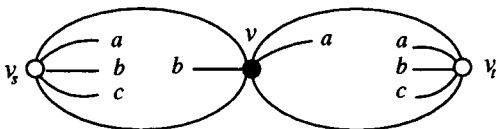


FIG. 8. Illustration for symbols a , b , and c .

The seven terms in the equation defining $\alpha_s(i, i_1, i_2)$ satisfy

$$\chi'(G_1, i_1) = s(\eta) + t(\eta) + i_1 + q(\eta), \quad (\text{A1})$$

$$\chi'(G_2, i_2) = s(\psi) + t(\psi) + i_2 + q(\psi), \quad (\text{A2})$$

$$d(v) = t(\eta) + i_1 + s(\psi) + i_2. \quad (\text{A3})$$

$$d(G) - i = s(\eta) + i_1 + t(\psi) + i_2 - i, \quad (\text{A4})$$

$$d(v) - i_1 - i_2 + i = t(\eta) + s(\psi) + i, \quad (\text{A5})$$

$$d(G_1) - i_1 + i_2 - i = s(\eta) + t(\eta) + i_1 + i_2 - i, \quad \text{and} \quad (\text{A6})$$

$$d(G_2) - i_2 + i_1 - i = s(\psi) + t(\psi) + i_2 + i_1 - i. \quad (\text{A7})$$

Although $\alpha_s(i, i_1, i_2)$ is equal to one of the seven terms, we give a proof only for the case $\alpha_s(i, i_1, i_2) = \chi'(G_1, i_1)$. Furthermore there are two subcases, $q(\eta) < i_2$ and $q(\eta) \geq i_2$, but we give a proof for the first subcase. The proofs for the other cases are omitted since they are similar to the case above.

We first alter φ to increase $r(\varphi)$ to i . Substituting (A1) and (A4) to $\chi'(G_1, i_1) \geq d(G) - i$ and using $q(\eta) < i_2$, we have $i \geq t(\psi) - t(\eta)$. Let

$$a = \min\{i, s(\eta), i_2, i - t(\psi) + t(\eta)\}, \quad (\text{A8})$$

$$b = \min\{i - a, i_1, t(\psi)\}, \quad \text{and} \quad (\text{A9})$$

$$c = i - a - b. \quad (\text{A10})$$

Then clearly $i = a + b + c$ and $a, b, c \geq 0$. Furthermore we claim that the following two equations hold:

$$a + c \leq s(\eta), \quad \text{and} \quad (\text{A11})$$

$$b + c \leq t(\psi). \quad (\text{A12})$$

One can verify (A11) as follows. Clearly

$$\begin{aligned} i &\leq h_s(i_1, i_2) = \min\{\delta_{st}(G), s(\eta) + t(\psi)\} \\ &= \min\{s(\eta) + i_1, t(\psi) + i_2, s(\eta) + t(\psi)\}. \end{aligned} \quad (\text{A13})$$

By (A9) and (A10) we have

$$\begin{aligned} a + c &= i - b \\ &= i - \min\{i - a, i_1, t(\psi)\} \\ &= \max\{a, i - i_1, i - t(\psi)\}. \end{aligned} \quad (\text{A14})$$

By (A8), (A13), and (A14) we have

$$\begin{aligned} a + c &\leq \max\{a, (s(\eta) + i_1) - i_1, (s(\eta) + t(\psi)) - t(\psi)\} \\ &= \max\{a, s(\eta)\} \\ &= s(\eta). \end{aligned}$$

Thus we have verified (A11). Similarly one can verify (A12). By (A8)–(A12) we can rename $i = a + b + c$ colors of the coloring ψ as follows: replace any a colors in $R(\psi)$ and any c colors in $T(\psi)$ with $a + c$ colors in $S(\eta)$, and replace any b colors among the remaining colors in $T(\psi)$ with b colors in $R(\eta)$. The resulting coloring φ of G extended from η and this altered ψ satisfies $r(\varphi) = i$.

We finally alter the coloring φ above to decrease $\#\varphi$ to $\alpha_s(i, i_1, i_2) = \#\eta$ without changing $r(\varphi)$. We claim that the following three inequalities hold:

$$q(\eta) \geq i_2 - a, \quad (\text{A15})$$

$$t(\eta) \geq t(\psi) - (b + c), \quad \text{and} \quad (\text{A16})$$

$$s(\eta) - (a + c) + q(\eta) - (i_2 - a) \geq s(\psi). \quad (\text{A17})$$

We now verify (A15). By (A8) we have

$$i_2 - a = \max\{i_2 - i, i_2 - s(\eta), 0, i_2 - i + t(\psi) - t(\eta)\}. \quad (\text{A18})$$

Since $\alpha_s(i, i_1, i_2) = \chi'(G_1, i_1)$, $\chi'(G_1, i_1) \geq d(G) - i$. Substitute (A1) and (A4) into $\chi'(G_1, i_1) \geq d(G) - i$; then we have

$$q(\eta) \geq i_2 - i + t(\psi) - t(\eta). \quad (\text{A19})$$

Similarly, using (A1), (A3), and (A6), we have

$$q(\eta) \geq i_2 - s(\eta), \quad \text{and} \quad (\text{A20})$$

$$q(\eta) \geq i_2 - i. \quad (\text{A21})$$

From (A18)–(A21) we have $q(\eta) \geq i_2 - a$. Thus we have verified (A15). Similarly one can verify (A16) and (A17). By the inequalities (A15)–(A17), we can alter η and ψ so that η and ψ , and hence their extension φ , use only the $\#\eta$ colors in $\eta(E(G_1))$ as follows:

- (1) replace the $i_2 - a$ remaining colors in $R(\psi)$ with colors in $Q(\eta)$;
- (2) replace the remaining $t(\psi) - (b + c)$ colors in $T(\psi)$ with colors in $T(\eta)$;
- (3) replace the colors in $S(\psi)$ with $s(\psi)$ colors among the remaining $s(\eta) - (a + c)$ colors in $S(\eta)$ plus the remaining $q(\eta) - (i_2 - a)$ colors in $Q(\eta)$; and finally
- (4) replace the colors in $Q(\psi)$ with colors in η which have not been used yet for G_2 .

Note that (4) above can be done since $\#\eta \geq \#\psi$. Then the resulting coloring φ satisfies $\#\varphi = \#\eta = \alpha_s(i, i_1, i_2)$ and $r(\varphi) = i$. Q.E.D.

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