Total Colorings of Degenerated Graphs

Shuji Isobe, Xiao Zhou, and Takao Nishizeki

Graduate School of Information Sciences, Tohoku University,
Aoba-yama 05, Sendai, 980-8579, Japan.
{iso@nishizeki.,zhou@,nishi@}ecei.tohoku.ac.jp

Abstract. A total coloring of a graph G is a coloring of all elements of G, i.e. vertices and edges, in such a way that no two adjacent or incident elements receive the same color. A graph G is s-degenerated for a positive integer s if G can be reduced to a trivial graph by successive removal of vertices with degree $\leq s$. We prove that an s-degenerated graph G has a total coloring with $\Delta + 1$ colors if the maximum degree Δ of G is sufficiently large, say $\Delta \geq 4s + 3$. Our proof yields an efficient algorithm to find such a total coloring. We also give a linear-time algorithm to find a total coloring of a graph G with the minimum number of colors if G is a partial k-tree, i.e. the tree-width of G is bounded by a fixed integer k.

1 Introduction

We deal with a total coloring of a *simple graph* G, which has no multiple edges or selfloops. A total coloring is a mixture of ordinary vertex-coloring and edge-coloring. That is, a *total coloring* of G is an assignment of colors to its vertices and edges so that no two adjacent vertices have the same color, no two adjacent edges have the same color, and no edge has the same color as one of its ends [14]. The minimum number of colors required for a total coloring of a graph G is called the *total chromatic number* of G, and denoted by $\chi_t(G)$. Figure 1(a) illustrates a total coloring of a graph G using $\chi_t(G) = 4$ colors. Let $\Delta(G)$ be the maximum degree of G, then clearly $\Delta(G) + 1 \leq \chi_t(G)$, and hence $\Delta(G) + 1$ is a lower bound on $\chi_t(G)$. On the other hand, it is conjectured that an upper bound $\chi_t(G) \leq \Delta(G) + 2$ holds for any simple graph G. However, this "total coloring conjecture" has not been verified [8,14]. The *total coloring problem* is to find a total coloring of a given graph G with the minimum number $\chi_t(G)$ of colors. Since the problem is NP-hard [11], it is very unlikely that there exists an efficient algorithm to solve the problem for general graphs. However, there would exist an efficient algorithm for a restricted class of graphs such as "s-degenerated graphs" and "partial k-trees" defined below.

A graph is said to be *s-degenerated* for an integer $s \geq 1$ if it can be reduced to a trivial graph by successive removal of vertices with degree $\leq s$. For example, the graph in Fig. 1(a) is 2-degenerated, and every planar graph is 5-degenerated. An s-degenerated graph has a favorable property on the vertex-coloring: let $\chi(G)$ be the *chromatic number* of a graph G, that is, the minimum number of colors required for a vertex-coloring of G, then clearly $\chi(G) \leq s + 1$ for any

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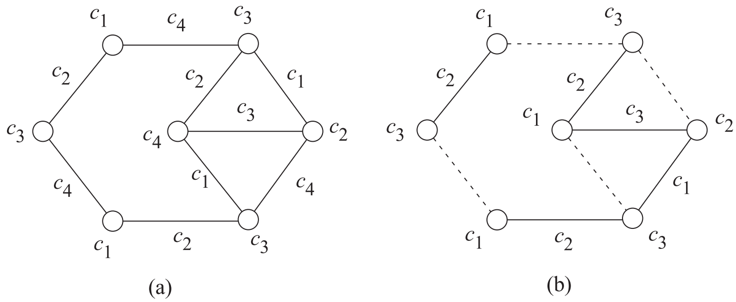

Fig. 1. (a) A total coloring of a graph G with $\chi_t(G) = 4$ colors c_1, c_2, c_3 and c_4, and (b) a semi-total coloring of G for $U = V$ and $F \subseteq E$ with $\chi(G; U, F) = 3$ colors c_1, c_2 and c_3.

s-degenerated graph G [6,8,12]. Let $\chi'(G)$ be the *chromatic index* of G, that is, the minimum number of colors required for an edge-coloring of G. Then clearly $\Delta(G) \le \chi'(G)$, and hence $\Delta(G)$ is a lower bound on $\chi'(G)$. An s-degenerated graph has a favorable property also on an edge-coloring: $\chi'(G) = \Delta(G)$ if G is an s-degenerated graph and $\Delta(G) \ge 2s$ [13]. Thus there is a simple sufficient condition on $\Delta(G)$, i.e. $\Delta(G) \ge 2s$, for the chromatic index $\chi'(G)$ of an s-degenerated graph G to be equal to the trivial lower bound $\Delta(G)$. However, it has not been known whether there is a simple sufficient condition on $\Delta(G)$ for the total chromatic number $\chi_t(G)$ to be equal to the trivial lower bound $\Delta(G) + 1$.

A graph with bounded tree-width k is called a *partial k-tree*; the formal definition of partial k-trees will be given in Section 2. Any partial k-tree is k-degenerated, but the converse is not always true. Many combinatorial problems can be efficiently solved for partial k-trees with bounded k [1,2,4]. In particular, both the vertex-coloring problem and the edge-coloring problem can be solved in linear time for partial k-trees [2,15]. However, no efficient algorithm has been known for the total coloring problem on partial k-trees. Although the total coloring problem can be solved in polynomial time for partial k-trees by a dynamic programming algorithm, the time complexity $O(n^{1+2^{4(k+1)}})$ is very high [7].

In this paper, we first present a sufficient condition on $\Delta(G)$ for the total chromatic number $\chi_t(G)$ of an s-degenerated graph G to be equal to the trivial lower bound $\Delta(G) + 1$: we prove our main theorem that $\chi_t(G) = \Delta(G) + 1$ if $\Delta(G) \ge 4s + 3$. The condition $\Delta(G) \ge 4s + 3$ for the total chromatic number is simple and interesting, compared to Vizing's condition $\Delta(G) \ge 2s$ for the chromatic index. Our proof immediately yields an efficient algorithm to find a total coloring of G with $\chi_t(G) = \Delta(G) + 1$ colors in time $O(sn^2)$ if $\Delta(G) \ge 4s + 3$, where n is the number of vertices in G. The complexity can be improved to $O(n \log n)$ in particular if $\Delta(G) \ge 6s + 1$ and $s = O(1)$. Hence the total coloring problem can be solved in time $O(n \log n)$ for a fairly large class of graphs including all planar graphs with sufficiently large maximum degree. We then show that one can find a total coloring of a given partial k-tree G with $\chi_t(G)$ colors in linear time and hence the total coloring problem can be solved in linear time for partial k-trees of bounded k.

2 Preliminaries

We denote by $G = (V, E)$ a simple undirected graph with a vertex set V and an edge set E. Let $n = |V|$ throughout the paper. We denote by $d(v, G)$ the *degree of a vertex v in G*, and by $\Delta(G)$ the *maximum degree of G*. For a set $F \subseteq E$, we denote by $G_F = (V, F)$ the spanning subgraph of G induced by the edge set F. A spanning subgraph of G is called a *forest* of G if each connected component is a tree. A forest of G is called a *linear forest* of G if each connected component is a single isolated vertex or a path. For example, $G_F = (V, F)$ is a linear forest of the graph G in Fig. 1(b) if F consists of the five edges drawn by solid lines.

One of the key ideas in the proof of our main theorem is to introduce a "semi-total coloring," which generalizes vertex-, edge- and total colorings. Let C be a set of colors, let U be a subset of V, and let F be a subset of E. Then a *semi-total coloring of a graph G for U and F* is a mapping $f : U \cup F \to C$ such that

(i) $f(v) \neq f(w)$ if $v, w \in U$ and $(v, w) \in E$;
(ii) $f(e) \neq f(e')$ if $e, e' \in F$ and e and e' share a common end; and
(iii) $f(v) \neq f(e)$ if $v \in U$, $e \in F$ and e is incident to v.

The semi-total coloring is sometimes called a partial total coloring [14]. The vertices in $\overline{U} = V - U$ and the edges in $\overline{F} = E - F$ are not colored by f. Figure 1(b) depicts a semi-total coloring of the graph G in Fig. 1(a) for $U = V$ and F, where $C = \{c_1, c_2, c_3\}$ and all edges in F are drawn by solid lines.

The minimum number of colors required for a semi-total coloring of G for U and F is called the *semi-total chromatic number of G for U and F*, and is denoted by $\chi(G; U, F)$. Then obviously $\Delta(G_F) + 1 \leq \chi(G; V, F)$, where $G_F = (V, F)$ is a spanning subgraph of G. Clearly, a total coloring of G is a semi-total coloring of G for $U = V$ and $F = E$; a vertex-coloring of G is a semi-total coloring of G for $U = V$ and $F = \emptyset$; and an edge-coloring of G is a semi-total coloring of G for $U = \emptyset$ and $F = E$.

Another idea is a "superimposing" of colorings. Suppose that g is a semi-total coloring of a graph $G = (V, E)$ for $U = V$ and $F \subseteq E$, h is an edge-coloring of $G_{\overline{F}} = (V, E - F)$, and g and h use no common color. Then, superimposing g on h, one can obtain a total coloring f of G, and hence

$$\chi_t(G) \leq \chi(G; V, F) + \chi'(G_{\overline{F}}). \tag{1}$$

Unfortunately, the total coloring f constructed from g and h may use more than $\chi_t(G)$ colors even if g uses the minimum number $\chi(G; V, F)$ of colors and h uses the minimum number $\chi'(G_{\overline{F}})$ of colors, because the equality in Eq. (1) does not always hold. For example, for the graph G in Fig. 1(b), $\chi_t(G) = 4$, $\chi(G; V, F) = 3$, $\chi'(G_{\overline{F}}) = 2$, and hence $\chi_t(G) < \chi(G; V, F) + \chi'(G_{\overline{F}})$. However, we will show in Section 3 as the main theorem that if G is an s-degenerated graph and $\Delta(G) \geq 4s+3$ then there is a subset $F \subset E$ such that the equality in Eq. (1) holds and $\chi(G;V,F) + \chi'(G_{\overline{F}}) = \Delta(G)+1$, that is,

$$\chi_{\mathrm{t}}(G) = \chi(G;V,F) + \chi'(G_{\overline{F}}) = \Delta(G)+1. \tag{2}$$

We will show in Section 4 that one can efficiently find a semi-total coloring g of G for V and F with $\chi(G;V,F)$ colors and an edge-coloring h of $G_{\overline{F}}$ with $\chi'(G_{\overline{F}})$ colors, and hence one can efficiently find a total coloring f of G with $\chi_{\mathrm{t}}(G)$ colors simply by superimposing g and h.

We now recursively define a k-tree: a graph $G=(V,E)$ is a *k-tree* if it is a complete graph of k vertices or it has a vertex $v \in V$ of degree k whose neighbors induce a clique of size k and the graph $G-\{v\}$ obtained from G by deleting the vertex v and all edges incident to v is again a k-tree. We then define a partial k-tree: a graph is a *partial k-tree* if it is a subgraph of a k-tree [1,2,15]. The graph in Fig. 1(a) is indeed a partial 3-tree. In this paper we assume that $k = O(1)$.

For an integer $s \geq 1$, a graph G is defined to be *s-degenerated* (or *s-inductive*) if G can be reduced to a trivial graph by the successive removal of vertices having degree at most s [6,8,16]. We do not assume that $s = O(1)$. By the definition of an s-degenerated graph $G=(V,E)$, there exists a numbering $\varphi : V \to \{1,2,\cdots,n\}$ such that any vertex $v \in V$ has at most s neighbors numbered by φ with integers larger than $\varphi(v)$, that is,

$$|\{x \in V : (v,x) \in E,\ \varphi(v) < \varphi(x)\}| \leq s.$$

Such a numbering φ is called an *s-numbering of G*. An s-numbering of any s-degenerated graph G can be found in linear time [9].

For a vertex v in a graph $G=(V,E)$ and a numbering $\varphi : V \to \{1,2,\cdots,n\}$, we write

$$\begin{aligned} E_{\varphi}^{\mathrm{fw}}(v,G) &= \{(v,x) \in E : \varphi(v) < \varphi(x)\}; \\ E_{\varphi}^{\mathrm{bw}}(v,G) &= \{(v,x) \in E : \varphi(v) > \varphi(x)\}; \\ d_{\varphi}^{\mathrm{fw}}(v,G) &= |E_{\varphi}^{\mathrm{fw}}(v,G)|; \text{ and} \\ d_{\varphi}^{\mathrm{bw}}(v,G) &= |E_{\varphi}^{\mathrm{bw}}(v,G)|. \end{aligned}$$

The edges in $E_{\varphi}^{\mathrm{fw}}(v,G)$ are called the *forward edges of* v, and those in $E_{\varphi}^{\mathrm{bw}}(v,G)$ the *backward edges of* v. Clearly $d(v,G) = d_{\varphi}^{\mathrm{fw}}(v,G) + d_{\varphi}^{\mathrm{bw}}(v,G)$. A graph G is s-degenerated if and only if there is a numbering φ such that $d_{\varphi}^{\mathrm{fw}}(v,G) \leq s$ for any vertex $v \in V$. An s-degenerated graph has the following two favorable properties on colorings, which have been mentioned in Introduction.

Lemma 1. *For any s-degenerated graph G, the following* (a) *and* (b) *hold:*

(a) $\chi(G) \leq s+1$ [6,8,9,12]; *and*
(b) *if* $\Delta(G) \geq 2s$, *then* $\chi'(G) = \Delta(G)$ [13,16].

3 Main Theorem

In this section we prove the following main theorem.

Theorem 1. *If G is an s-degenerated graph and $\Delta(G) \geq 4s+3$, then* $\chi_t(G) = \Delta(G) + 1$.

A result by Borodin *et al.* on a "total *list* coloring" [3, Theorem 7] implies that $\chi_t(G) = \Delta(G) + 1$ if G is an s-degenerated graph and $\Delta(G) \geq 2s^2$. Theorem 1 is better than these results.

We show in the remainder of this section that there is a subset $F \subset E$ satisfying Eq. (2). F will be found as a union of $s+1$ edge-disjoint linear forests of G.

We first show in the following lemma that G can be decomposed to $s+1$ edge-disjoint forests.

Lemma 2. *If $G = (V, E)$ is an s-degenerated graph, then there exists a partition $\{F_1, F_2, \cdots, F_{s+1}\}$ of E such that, for any index $j \in \{1, 2, \cdots, s+1\}$,*

(a) *G_{F_j} is a forest; and*
(b) *$d(v, G_{F_j}) \geq 3$ if $d(v, G) \geq 4s+3$.*

Proof. Let $G = (V, E)$ be an s-degenerated graph, and let $\varphi : V \to \{1, 2, \cdots, n\}$ be an s-numbering of G.

We first find $F_1, F_2, \cdots, F_{s+1}$. Construct a new graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ from G as follows: for any vertex $v \in V$,

- let $t = \lceil d_\varphi^{\mathrm{bw}}(v, G)/(s+1) \rceil$, and replace v with its $t+1$ copies v_{fw}, $v_{\mathrm{bw}}^1, v_{\mathrm{bw}}^2, \cdots, v_{\mathrm{bw}}^t$;
- attach all forward edges in $E_\varphi^{\mathrm{fw}}(v, G)$ to the copy v_{fw}; and
- let $\{E_{\mathrm{bw}}^1, E_{\mathrm{bw}}^2, \cdots, E_{\mathrm{bw}}^t\}$ be any partition of the set $E_\varphi^{\mathrm{bw}}(v, G)$ of backward edges such that

$$|E_{\mathrm{bw}}^i| \begin{cases} = s+1 \text{ if } 1 \leq i \leq t-1; \\ \leq s+1 \text{ if } i = t, \end{cases} \tag{3}$$

  and attach all edges in E_{bw}^i to the copy v_{bw}^i for each $i = 1, 2, \cdots, t$.

Clearly, $\widetilde{G}$ is bipartite. Since φ is an s-numbering of G, $d(v_{\mathrm{fw}}, \widetilde{G}) = d_\varphi^{\mathrm{fw}}(v, G) \leq s$ for any vertex $v \in V$. By Eq. (3) $d(v_{\mathrm{bw}}^i, \widetilde{G}) \leq s+1$ for any vertex $v \in V$ and any index $i \in \{1, 2, \cdots, t\}$. Thus we have $\Delta(\widetilde{G}) \leq s+1$.

Since $\widetilde{G}$ is bipartite and $\Delta(\widetilde{G}) \leq s+1$, König's theorem implies that $\chi'(\widetilde{G}) = \Delta(\widetilde{G}) \leq s+1$ [6,8], and hence $\widetilde{G}$ has an edge-coloring $f : \widetilde{E} \to C$ for a set $C = \{c_1, c_2, \cdots, c_{s+1}\}$ of $s+1$ colors. For each color $c_j \in C$, let $\widetilde{F}_j$ be the color class of c_j, that is, $\widetilde{F}_j = \{e \in \widetilde{E} : f(e) = c_j\}$, and let F_j be the set of edges in E corresponding to $\widetilde{F}_j$. Since f is an edge-coloring of $\widetilde{G}$, $\{\widetilde{F}_1, \widetilde{F}_2, \cdots, \widetilde{F}_{s+1}\}$ is a partition of $\widetilde{E}$, and hence $\{F_1, F_2, \cdots, F_{s+1}\}$ is a partition of E. Thus we have found $F_1, F_2, \cdots, F_{s+1}$.

We then prove that F_j found as above satisfies (a) and (b) for any index $j \in \{1, 2, \cdots, s+1\}$.

(a) It suffices to prove that the s-numbering φ of G is indeed a 1-numbering of G_{F_j}, that is, $d^{\rm fw}_{\varphi}(v, G_{F_j}) \le 1$ for any vertex $v \in V$. By the construction of $\widetilde{G}$, all forward edges of v in G are attached to the copy $v_{\rm fw}$ in $\widetilde{G}$. At most one of them is colored with c_j by f since f is an edge-coloring of $\widetilde{G}$. Thus $\widetilde{F}_j$ contains at most one edge incident to $v_{\rm fw}$, and hence we have $d^{\rm fw}_{\varphi}(v, G_{F_j}) \le 1$.

(b) Let v be any vertex in V with $d(v, G) \ge 4s + 3$.

We first claim that $d(v^i_{\rm bw}, \widetilde{G}) = s+1$ for each $i \in \{1, 2, 3\}$. By the construction of $\widetilde{G}$, $d(v^i_{\rm bw}, \widetilde{G}) = s + 1$ if $i \le \lfloor d^{\rm bw}_{\varphi}(v, G)/(s+1) \rfloor$. It therefore suffices to prove that $\lfloor d^{\rm bw}_{\varphi}(v, G)/(s+1) \rfloor \ge 3$. Clearly, $d^{\rm fw}_{\varphi}(v, G) \le s$ and $d(v, G) = d^{\rm fw}_{\varphi}(v, G) + d^{\rm bw}_{\varphi}(v, G)$. Hence we have

$$\left\lfloor \frac{d^{\rm bw}_{\varphi}(v, G)}{s+1} \right\rfloor = \left\lfloor \frac{d(v, G) - d^{\rm fw}_{\varphi}(v, G)}{s+1} \right\rfloor \ge \left\lfloor \frac{4s+3-s}{s+1} \right\rfloor = 3. \tag{4}$$

Thus we have proved the claim.

We then prove that $d(v, G_{F_j}) \ge 3$. The edge-coloring f of $\widetilde{G}$ uses exactly $s+1$ colors in C, and $d(v^i_{\rm bw}, \widetilde{G}) = s+1$ for each $i \in \{1, 2, 3\}$. Therefore exactly one of the edges incident to $v^i_{\rm bw}$ in $\widetilde{G}$ is colored with $c_j \in C$ by f. We thus have $d(v, G_{F_j}) \ge 3$. □

One can construct a linear forest from a forest as in the following lemma.

Lemma 3. *Let $T = (V, F)$ be a forest, let $S = \{v \in V : d(v, T) \ge 3\}$, and let U be any subset of S. Then T has a linear forest $T_L = (V, L)$, $L \subseteq F$, such that every vertex in U is an end of a path in T_L, and every vertex in $S - U$ is an interior vertex of a path in T_L, that is,*

$$d(v, T_L) = \begin{cases} 1 & \text{if } v \in U; \text{ and} \\ 2 & \text{if } v \in S - U. \end{cases} \tag{5}$$

Furthermore L can be found in linear time.

Sketchy Proof. Regard each tree in forest T as a rooted tree. Then one can easily find a linear forest of the tree by the breadth-first search. □

By Lemmas 2 and 3 one can find $s+1$ linear forests $G_{L_1}, G_{L_2}, \cdots, G_{L_{s+1}}$ of G as in the following lemma.

Lemma 4. *If $G = (V, E)$ is an s-degenerated graph and $\Delta(G) \ge 4s + 3$, then for any partition $\{U_1, U_2, \cdots, U_{s+1}\}$ of V there exist mutually disjoint subsets $L_1, L_2, \cdots, L_{s+1}$ of E such that*

(a) *for each $j \in \{1, 2, \cdots, s+1\}$, G_{L_j} is a linear forest, and*

$$d(v, G_{L_j}) \le \begin{cases} 1 & \text{if } v \in U_j; \\ 2 & \text{if } v \in V - U_j; \end{cases} \tag{6}$$

(b) $\Delta(G_F) = 2s+1$, *where* $F = L_1 \cup L_2 \cup \cdots \cup L_{s+1}$; *and*
(c) $\Delta(G_F) + \Delta(G_{\overline{F}}) = \Delta(G)$, *where* $\overline{F} = E - F$.

Proof. Let $G = (V, E)$ be an s-degenerated graph, and let $\Delta(G) \geq 4s+3$. We find $L_1, L_2, \cdots, L_{s+1}$ as follows.

We first construct a new graph $G^* = (V^*, E^*)$ from G as follows: for each vertex $v \in V$ with $d(v, G) < 4s+3$, add $(4s+3) - d(v, G)$ dummy vertices and join each of them with v by a dummy edge. Clearly

$$d(v, G^*) = \begin{cases} d(v, G) & \text{if } v \in V \text{ and } d(v, G) \geq 4s+3; \\ 4s+3 & \text{if } v \in V \text{ and } d(v, G) < 4s+3; \text{ and} \\ 1 & \text{if } v \in V^* - V, \end{cases} \tag{7}$$

and hence

$$V = \{v \in V^* : d(v, G^*) \geq 4s+3\}. \tag{8}$$

Since $\Delta(G) \geq 4s+3$,

$$\Delta(G^*) = \Delta(G). \tag{9}$$

We then find $s+1$ forests of G^*. Since G is s-degenerated, G^* is also s-degenerated. Therefore, applying Lemma 2 to G^*, one can know that there exists a partition $\{F_1, F_2, \cdots, F_{s+1}\}$ of E^* such that, for any index $j \in \{1, 2, \cdots, s+1\}$,

(i) $G^*_{F_j}$ is a forest; and
(ii) $d(v, G^*_{F_j}) \geq 3$ if $d(v, G^*) \geq 4s+3$.

By (ii) and Eqs. (7) and (8) we have $V = \{v \in V^* : d(v, G^*_{F_j}) \geq 3\}$.

We then find $s+1$ linear forests of G^*. Let $\{U_1, U_2, \cdots, U_{s+1}\}$ be any partition of V. For each $j \in \{1, 2, \cdots, s+1\}$, apply Lemma 3 to $T = G^*_{F_j}$, $S = V = \{v \in V^* : d(v, G^*_{F_j}) \geq 3\}$ and $U = U_j \subseteq V$, then one can know that the forest $G^*_{F_j}$ has a linear forest $G^*_{L^*_j} = (V^*, L^*_j)$ such that

$$d(v, G^*_{L^*_j}) = \begin{cases} 1 \text{ if } v \in U_j; \text{ and} \\ 2 \text{ if } v \in V - U_j. \end{cases} \tag{10}$$

Since $L^*_j \subseteq F_j$, $1 \leq j \leq s+1$, and $F_1, F_2, \cdots, F_{s+1}$ are mutually disjoint with each other, $L^*_1, L^*_2, \cdots, L^*_{s+1}$ are also mutually disjoint with each other.

We then find $L_1, L_2, \cdots, L_{s+1}$ from $L^*_1, L^*_2, \cdots, L^*_{s+1}$; for each $j \in \{1, 2, \cdots, s+1\}$, let L_j be the set of all non-dummy edges in L^*_j, that is, $L_j = L^*_j \cap E$. Then $L_j \subseteq L^*_j$. Furthermore, one can easily observe that

$$d(v, G_{L_j}) \begin{cases} \leq d(v, G^*_{L^*_j}) \text{ if } v \in V; \\ = d(v, G^*_{L^*_j}) \text{ if } v \in V \text{ and } d(v, G) \geq 4s+3. \end{cases} \tag{11}$$

Since $L^*_1, L^*_2, \cdots, L^*_{s+1}$ are mutually disjoint with each other, $L_1, L_2, \cdots, L_{s+1}$ are also mutually disjoint with each other. Thus we have found $L_1, L_2, \cdots, L_{s+1}$.

We shall prove that $L_1, L_2, \cdots, L_{s+1}$ and $F = L_1 \cup L_2 \cup \cdots \cup L_{s+1}$ satisfy (a)–(c).

(a) Since $L_j \subseteq L_j^*$ and $G^*_{L_j^*}$ is a linear forest of G^*, G_{L_j} is a linear forest of G. Let v be any vertex in V. If $v \in U_j$, then by Eqs. (10) and (11) we have $d(v, G_{L_j}) \leq d(v, G^*_{L_j^*}) = 1$. Similarly, if $v \in V - U_j$, then we have $d(v, G_{L_j}) \leq d(v, G^*_{L_j^*}) = 2$.

(b) We first prove that $\Delta(G_F) \leq 2s+1$. Let $F^* = L_1^* \cup L_2^* \cup \cdots \cup L_{s+1}^*$. Let v be any vertex in V, and let j be the index such that $v \in U_j$. Then by Eq. (10) we have

$$d(v, G^*_{F^*}) = 2s + 1. \tag{12}$$

Since $F \subseteq F^*$, we have $d(v, G_F) \leq d(v, G^*_{F^*}) = 2s+1$. Thus we have $\Delta(G_F) \leq 2s+1$.

We then prove that $\Delta(G_F) \geq 2s+1$. Since $\Delta(G) \geq 4s+3$, G has a vertex v with $d(v, G) \geq 4s+3$. Since $F = L_1 \cup L_2 \cup \cdots \cup L_{s+1}$, we have

$$d(v, G_F) = \sum_{i=1}^{s+1} d(v, G_{L_i}). \tag{13}$$

By Eqs. (11), (12) and (13) we have $d(v, G_F) = d(v, G^*_{F^*}) = 2s+1$. We thus have $2s+1 \leq \Delta(G_F)$.

(c) Clearly, $\Delta(G_F) + \Delta(G_{\overline{F}}) \geq \Delta(G)$ for any set $F \subseteq E$. We shall therefore prove that $\Delta(G_F) + \Delta(G_{\overline{F}}) \leq \Delta(G)$, that is, $\Delta(G_{\overline{F}}) \leq \Delta(G) - \Delta(G_F)$. Since $E \subseteq E^*$ and $F = F^* \cap E$,

$$E - F \subseteq E^* - F^*. \tag{14}$$

For any vertex $v \in V$, by (b) above and Eqs. (9), (12) and (14) we have

$$\begin{aligned} d(v, G_{\overline{F}}) &= d(v, G_{E-F}) \\ &\leq d(v, G^*_{E^*-F^*}) \\ &= d(v, G^*) - d(v, G^*_{F^*}) \\ &\leq \Delta(G^*) - (2s+1) \\ &= \Delta(G) - \Delta(G_F). \end{aligned}$$

Thus we have $\Delta(G_{\overline{F}}) \leq \Delta(G) - \Delta(G_F)$. □

By Lemma 1(a) any s-degenerated graph G has a vertex-coloring with $s+1$ colors. Choose the set of color classes as the partition $\{U_1, U_2, \cdots, U_{s+1}\}$ in Lemma 4. Then there is a subset $F \subset E$ satisfying Eq. (2), as shown in the following theorem.

Theorem 2. *If $G = (V, E)$ is an s-degenerated graph and $\Delta(G) \geq 4s+3$, then there exists a subset F of E such that*

(a) $\chi(G; V, F) = \Delta(G_F) + 1$;
(b) $\chi'(G_{\overline{F}}) = \Delta(G_{\overline{F}})$, *where* $\overline{F} = E - F$;
(c) $\Delta(G_F) + \Delta(G_{\overline{F}}) = \Delta(G)$; *and*
(d) $\chi_t(G) = \chi(G; V, F) + \chi'(G_{\overline{F}}) = \Delta(G) + 1.$

Proof. Let $G = (V, E)$ be an s-degenerated graph, and let $\Delta(G) \geq 4s+3$. Since G is s-degenerated, by Lemma 1(a) $\chi(G) \leq s+1$ and hence G has a vertex-coloring $f : V \to C$ for a set $C = \{c_1, c_2, \cdots, c_{s+1}\}$ of $s+1$ colors. For each color $c_j \in C$, let U_j be the color class of c_j, that is, $U_j = \{v \in V : f(v) = c_j\}$. Then $\{U_1, U_2, \cdots, U_{s+1}\}$ is a partition of V and each of $U_1, U_2, \cdots, U_{s+1}$ is an independent set of G. By Lemma 4 for the partition $\{U_1, U_2, \cdots, U_{s+1}\}$ there exist mutually disjoint subsets $L_1, L_2, \cdots, L_{s+1}$ of E satisfying the conditions (a)–(c) in Lemma 4. Since the condition (c) in Theorem 2 is the same as (c) in Lemma 4, we shall show that $F = L_1 \cup L_2 \cup \cdots \cup L_{s+1}$ satisfies the conditions (a), (b) and (d) in Theorem 2.

(a) Clearly, $\chi(G; V, F) \geq \Delta(G_F) + 1$ for any set $F \subseteq E$. Therefore it suffices to prove that $\chi(G; V, F) \leq \Delta(G_F) + 1$.

For each $j \in \{1, 2, \cdots, s+1\}$, we first construct a semi-total coloring g_j of G for U_j and L_j with two colors, as follows. Since G_{L_j} is a linear forest by Lemma 4(a), we have $\chi'(G_{L_j}) = \Delta(G_{L_j}) \leq 2$ and hence G_{L_j} has an edge-coloring $f_j : L_j \to C_j$ for a set C_j of two colors. Since $|C_j| = 2$ and $d(v, G_{L_j}) \leq 1$ for each vertex $v \in U_j$ by Eq. (6), there is a color $c_v \in C_j$ such that the edge-coloring f_j does not assign c_v to any edge incindet to v. We then extend the mapping $f_j : L_j \to C_j$ to a mapping $g_j : U_j \cup L_j \to C_j$, as follows:

$$g_j(x) = \begin{cases} f_j(x) & \text{for each edge } x \in L_j\text{; and} \\ c_x & \text{for each vertex } x \in U_j. \end{cases} \tag{15}$$

Then g_j is a semi-total coloring of G for U_j and L_j, that is, g_j satisfies the three conditions (i), (ii) and (iii) on a semi-total coloring mentioned in Section 2, as follows. Since U_j is an independent set of G, clearly g_j satisfies the condition (i) for $U = U_j$ and $F = L_j$. Since f_j is an edge-coloring of G_{L_j}, g_j satisfies the condition (ii). By Eq. (15) g_j satisfies the condition (iii).

From $g_1, g_2, \cdots, g_{s+1}$ above we then construct a semi-total coloring g of G for V and F with $\Delta(G_F) + 1$ colors, as follows. One may assume that any two of the semi-total colorings $g_1, g_2, \cdots, g_{s+1}$ use no common color, that is, $C_p \cap C_q = \emptyset$ for any p and q, $1 \leq p < q \leq s+1$. Superimpose $g_1, g_2, \cdots, g_{s+1}$, and let $g : V \cup F \to C_1 \cup C_2 \cup \cdots \cup C_{s+1}$ be the resulting coloring. Then one can easily observe that g is a semi-total coloring of G for V and F. The semi-total coloring g uses $\sum_{i=1}^{s+1} |C_i| = 2(s+1)$ colors, and $\Delta(G_F) = 2s+1$ by Lemma 4(b). Therefore we have $\chi(G; V, F) \leq 2(s+1) = \Delta(G_F) + 1$.

(b) Since G is s-degenerated, the subgraph $G_{\overline{F}}$ of G is also s-degenerated. Since $\Delta(G) \geq 4s+3$, by the conditions (b) and (c) in Lemma 4 we have $\Delta(G_{\overline{F}}) = \Delta(G) - \Delta(G_F) \geq (4s+3) - (2s+1) = 2s+2 > 2s$. Therefore, by Lemma 1(b) we have $\chi'(G_{\overline{F}}) = \Delta(G_{\overline{F}})$.

(d) By (a), (b) and (c) we have $\chi(G; V, F) + \chi'(G_{\overline{F}}) = \Delta(G_F) + 1 + \Delta(G_{\overline{F}}) = \Delta(G) + 1$. Thus we shall prove that $\chi_t(G) = \chi(G; V, F) + \chi'(G_{\overline{F}})$.

By Eq. (1) $\chi_t(G) \leq \chi(G; V, F) + \chi'(G_{\overline{F}})$. Clearly $\chi_t(G) \geq \Delta(G) + 1 = \chi(G; V, F) + \chi'(G_{\overline{F}})$. Thus we have $\chi_t(G) = \chi(G; V, F) + \chi'(G_{\overline{F}}) = \Delta(G) + 1$. □

Theorem 2(d) implies Theorem 1.

4 Algorithms

From the proofs of Lemmas 2–4 and Theorem 2, one can know that the following algorithm correctly finds a total coloring of an s-degenerated graph $G = (V, E)$ with $\Delta(G) + 1$ colors if $\Delta(G) \geq 4s + 3$.

[**Total-Coloring Algorithm**]

Step 1. Find a vertex-coloring of a given s-degenerated graph G with $s + 1$ colors, and let $\{U_1, U_2, \cdots, U_{s+1}\}$ be the set of color classes.

Step 2. Construct a graph $G^* = (V^*, E^*)$ from G by adding dummy vertices and edges, as in the proof of Lemma 4.

Step 3. Construct a bipartite graph $\widetilde{G^*} = (\widetilde{V^*}, \widetilde{E^*})$ from G^* by splitting each vertex in G^*, as in the proof of Lemma 2. Note that $\Delta(\widetilde{G^*}) \leq s + 1$.

Step 4. Find an edge-coloring of the bipartite graph $\widetilde{G^*}$ with $s + 1$ colors, and let $\{\widetilde{F_1^*}, \widetilde{F_2^*}, \cdots, \widetilde{F_{s+1}^*}\}$ be the set of color classes. Let $\{F_1, F_2, \cdots, F_{s+1}\}$ be the partition of E^* corresponding to $\{\widetilde{F_1^*}, \widetilde{F_2^*}, \cdots, \widetilde{F_{s+1}^*}\}$, where $G^*_{F_j}$, $1 \leq j \leq s + 1$, is a forest of G^*.

Step 5. From each forest $G^*_{F_j}$ of G^*, $1 \leq j \leq s + 1$, find a linear forest $G^*_{L_j^*}$ of G^* such that

$$d(v, G^*_{L_j^*}) = \begin{cases} 1 \text{ if } v \in U_j; \\ 2 \text{ if } v \in V - U_j, \end{cases}$$

as in the proof of Lemma 3, where U_j is a color class found in Step 1.

Step 6. From each linear forest $G^*_{L_j^*}$ of G^*, obtain a linear forest G_{L_j} of G such that

$$d(v, G_{L_j}) \leq \begin{cases} 1 \text{ if } v \in U_j; \\ 2 \text{ if } v \in V - U_j, \end{cases}$$

by deleting all dummy vertices and edges as in the proof of Lemma 4.

Step 7. For each j, find an edge-coloring f_j of the linear forest G_{L_j} with two colors, and extend f_j to a semi-total coloring g_j of G for U_j and L_j as in the proof of Theorem 2.

Step 8. Superimposing $g_1, g_2, \cdots, g_{s+1}$, obtain a semi-total coloring g of G for V and $F = L_1 \cup L_2 \cup \cdots \cup L_{s+1}$ with $\Delta(G_F) + 1$ colors, as in the proof of Theorem 2.

Step 9. Find an edge-coloring h of $G_{\overline{F}}$ with $\Delta(G_{\overline{F}})$ colors.

Step 10. Superimposing g and h, obtain a total coloring of G with $\Delta(G) + 1$ colors.

We then show that all steps can be done in time $O(sn^2)$, using an algorithm for edge-coloring bipartite graphs [5] and an algorithm for edge-coloring s-degenerated graphs [15,16].

One can easily find the vertex-coloring of G in time $O(sn)$ by a simple greedy algorithm based on an s-numbering of G [6,8,9,12]. Note that G has at most sn edges. Thus Step 1 can be done in time $O(sn)$.

By the construction of the graph G^*, we have $|V^* - V| = |E^* - E| \leq (4s+3)n$ and hence

$$|V^*| \leq n + (4s+3)n = 4(s+1)n, \text{ and}$$
$$|E^*| \leq sn + (4s+3)n = (5s+3)n.$$

Thus one can construct the graph G^* in time $O(sn)$, and hence Step 2 can be done in time $O(sn)$.

Clearly

$$|\widetilde{E^*}| = |E^*| \leq (5s+3)n, \text{ and}$$
$$|\widetilde{V^*}| \leq 2|\widetilde{E^*}| \leq 2(5s+3)n.$$

Therefore one can construct $\widetilde{G^*}$ from G^* in time $O(sn)$. Thus Step 3 can be done in time $O(sn)$.

Since $\widetilde{G^*}$ is bipartite and $\Delta(\widetilde{G^*}) \leq s+1$, one can find the edge-coloring of $\widetilde{G^*}$ in time $O(|\widetilde{E^*}| \log \Delta(\widetilde{G^*}) = O(sn \log s)$ [5]. Note that $s \leq n$. Thus Step 4 can be done in time $O(sn \log n)$.

By Lemma 2, for each forest $G^*_{F_j}$, $1 \leq j \leq s+1$, one can find the linear forest $G^*_{L^*_j}$ in time $O(|V^*|) = O(sn)$. Therefore the $s+1$ linear forests $G^*_{L^*_1}, G^*_{L^*_2}, \cdots, G^*_{L^*_{s+1}}$ can be found in time $O((s+1)sn) = O(s^2 n)$. Thus Step 5 can be done in time $O(s^2 n)$.

From each linear forest $G^*_{L^*_j}$, $1 \leq j \leq s+1$, one can obtain the linear forest G_{L_j} in time $O(|L^*_j|) = O(sn)$ simply by deleting dummy vertices and edges. Therefore one can obtain the $s+1$ linear forests $G_{L_1}, G_{L_2}, \cdots, G_{L_{s+1}}$ in time $O((s+1)sn) = O(s^2 n)$. Thus Step 6 can be done in time $O(s^2 n)$.

For each j, $1 \leq j \leq s+1$, one can easily find an edge-coloring f_j of the linear forest G_{L_j} with two colors in time $O(|L_j|) = O(n)$, and can extend f_j to the semi-total coloring g_j in time $O(n)$. Therefore Step 7 can be done in time $O(sn)$.

Superimposing $g_1, g_2, \cdots, g_{s+1}$, one can obtain the semi-total coloring g of G for V and F in time $O(sn)$. Thus Step 8 can be done in time $O(sn)$.

Since $G_{\overline{F}}$ is s-degenerated, one can find the edge-coloring h of $G_{\overline{F}}$ in time $O(sn^2)$ [10, 15 p.604, 16 p.8]. Therefore Step 9 can be done in time $O(sn^2)$.

Superimposing g and h, one can obtain a total coloring of G with $\chi_t(G) = \Delta(G) + 1$ colors in time $O(sn)$. Thus Step 10 can be done in time $O(sn)$.

Thus all Steps 1–10 above can be done in time $O(sn^2)$, and hence we have the following theorem.

Theorem 3. *A total coloring of an s-degenerated graph G using $\chi_t(G) = \Delta(G) + 1$ colors can be found in time $O(sn^2)$ if $\Delta(G) \geq 4s+3$.*

The complexity $O(sn^2)$ can be improved as in the following two theorems.

Theorem 4. *A total coloring of an s-degenerated graph G using $\chi_t(G) = \Delta(G) + 1$ colors can be found in time $O(n \log n)$ if $\Delta(G) \geq 6s+1$ and $s = O(1)$.*

Sketchy Proof. Use an $O(n \log n)$ algorithm in [16] to find an edge-coloring of $G_{\overline{F}}$. □

Theorem 5. *The total coloring problem can be solved in linear time for partial k-trees G with bounded k.*

Sketchy Proof. For the case where $\Delta(G) < 4k + 3$, use the algorithm in [7] to find a total coloring of G in time $O(n\chi_t^{2^{4(k+1)}}) = O(n)$. For the case where $\Delta(G) \geq 4k + 3$, use our algorithm to find a total coloring of G, but use a linear-time algorithm in [15] to find an edge-coloring of $G_{\overline{F}}$ in Step 9. □

References

1. S. Arnborg and J. Lagergren, Easy problems for tree-decomposable graphs, J. Algorithms, 12(2), pp. 308–340, 1991.
2. H. L. Bodlaender, Polynomial algorithms for graph isomorphism and chromatic index on partial k-trees, J. Algorithms, 11(4), pp. 631–643, 1990.
3. O. V. Borodin, A. V. Kostochka and D. R. Woodall, List edge and list total colourings of multigraphs, J. Combinatorial Theory, Series B, 71, pp. 184–204, 1997.
4. R. B. Borie, R. G. Parker and C. A. Tovey, Automatic generation of linear-time algorithms from predicate calculus descriptions of problems on recursively constructed graph families, Algorithmica, 7, pp. 555–581, 1992.
5. R. Cole, K. Ost and S. Schirra, Edge-coloring bipartite multigraphs in $O(E \log D)$ time, Combinatorica, 21, pp. 5–12, 2001.
6. R. Diestel, Graph Theory, Springer, New York, 1997.
7. S. Isobe, X. Zhou and T. Nishizeki, A polynomial-time algorithm for finding total colorings of partial k-trees, Int. J. Found. Comput. Sci., 10(2), pp. 171–194, 1999.
8. T. R. Jensen and B. Toft, Graph Coloring Problems, John Wiley & Sons, New York, 1995.
9. D. Matula and L. Beck, Smallest-last ordering and clustering and graph coloring algorithms, J. Assoc. Comput. Mach., 30, pp. 417–427, 1983.
10. T. Nishizeki and N. Chiba, Planar Graphs: Theory and Algorithms, North-Holland, Amsterdam, 1988.
11. A. Sánchez-Arroyo. Determining the total colouring number is NP-hard, Discrete Math., 78, pp. 315–319, 1989.
12. G. Szekeres and H. Wilf, An inequality for the chromatic number of a graph, J. Combinatorial Theory, 4, pp. 1–3, 1968.
13. V. G. Vizing, Critical graphs with given chromatic class (in Russian), Metody Discret Analiz., 5, pp. 9–17, 1965.
14. H. P. Yap, Total Colourings of Graphs, Lect. Notes in Math., 1623, Springer, Berlin, 1996.
15. X. Zhou, S. Nakano and T. Nishizeki, Edge-coloring partial k-trees, J. Algorithms, 21, pp. 598–617, 1996.
16. X. Zhou and T. Nishizeki, Edge-coloring and f-coloring for various classes of graphs, J. Graph Algorithms and Applications, `http://www.cs.brown.edu/publications/jgaa/`, 3(1), pp. 1–18, 1999.