This is the appexdix for the paper "Propagation of Spontaneously Actuated Pulsive Vibration in Human Heart Wall and *In Vivo* Viscoelasticity Estimation" by Hiroshi Kanai published in *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control.* Vol. 51, No. 11, pp. 1931–1942 (November 2005)

APPENDIX I

THEORETICAL EQUATIONS OF A LAMB WAVE PROPAGATING ALONG VISCOELASTIC PLATE IMMERSED IN BLOOD

For the asymmetric Lamb wave in **Fig. 1**, let us assume that the Lamb wave propagates in the positive x-direction along the plate. The potential ϕ of the primary (longitudinal) wave and the potential in plate, ψ, of the SV wave are respectively given by

$$\phi = A\sinh(\eta y)\exp(jk_\mathrm{L}x) \tag{A.1}$$

$$\psi = B\cosh(\beta y)\exp(jk_\mathrm{L}x), \tag{A.2}$$

where A and B are amplitude constants, $j = \sqrt{-1}$, and k_L is the wave number. Using the wave numbers k_p for the primary wave and k_s for the secondary wave of the plate material, the

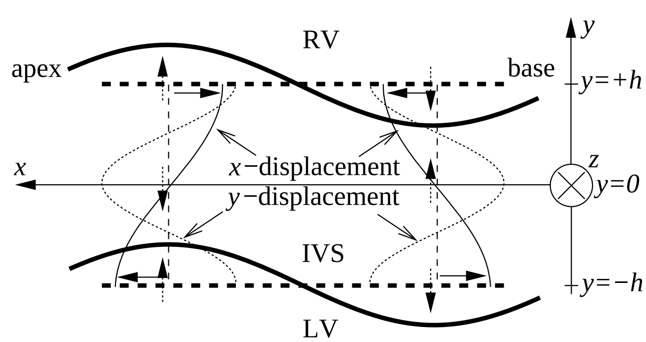

Fig. 1. Lamb wave with asymmetric mode of plate waves in the viscoelastic plate with thickness $2h$. The SV-wave component (y-displacement) and longitudinal component (x-displacement) are coupled, and the Lamb wave then propagates along the x-direction. Though a slightly higher order mode is illustrated, the lowest mode is probably dominant in actual vibration in the IVS.

following η and β are defined as

$$\eta = \sqrt{k_{\rm L}^2 - k_{\rm p}^2} \text{ [rad/m]} \tag{A.3}$$

$$\beta = \sqrt{k_{\rm L}^2 - k_{\rm s}^2} \text{ [rad/m].} \tag{A.4}$$

Let us assume that the myocardium is isotropic. Using the Lamb wave phase velocity $c_{\rm L}$, the primary wave speed $c_{\rm p}$, the secondary wave speed $c_{\rm s}$, Lamé elastic constants λ and μ, and the myocardial density $\rho_{\rm m}$, the wave numbers $k_{\rm L}$, $k_{\rm p}$, and $k_{\rm s}$ are described by

$$k_{\rm L} = \frac{\omega}{c_{\rm L}} \text{ [rad/m]} \tag{A.5}$$

$$k_{\rm p} = \frac{\omega}{c_{\rm p}} = \omega\sqrt{\frac{\rho_{\rm m}}{\lambda + 2\mu}} \text{ [rad/m]} \tag{A.6}$$

$$k_{\rm s} = \frac{\omega}{c_{\rm s}} = \omega\sqrt{\frac{\rho_{\rm m}}{\mu}} \text{ [rad/m],} \tag{A.7}$$

where $\omega = 2\pi f$ denotes the angular frequency. The displacement u_x in the x-direction and u_y in the y-direction are thus given by

$$u_x \equiv \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y} = \Big\{jk_{\rm L}A\sinh(\eta y) + \beta B\sinh(\beta y)\Big\}\exp(jk_{\rm L}x) \text{ [m]} \tag{A.8}$$

$$u_y \equiv \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x} = \Big\{\eta A\cosh(\eta y) - jk_{\rm L}B\cosh(\beta y)\Big\}\exp(jk_{\rm L}x) \text{ [m].} \tag{A.9}$$

From the stress-strain relationship with Lamé constants λ and μ for isotropic material, the stress σ_{yy} normal to y-axis is given by

$$\begin{aligned}
\sigma_{yy} &\equiv (\lambda + 2\mu)\frac{\partial u_y}{\partial y} + \lambda\frac{\partial u_x}{\partial x} \text{ [Pa]} \\
&= (\lambda + 2\mu)\left(\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\partial y}\right) + \lambda\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\psi}{\partial x\partial y}\right) \\
&= (\lambda + 2\mu)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) - 2\mu\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\psi}{\partial x\partial y}\right) \\
&= \mu\left\{\kappa^2\nabla^2\phi - 2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\psi}{\partial x\partial y}\right)\right\},
\end{aligned} \tag{A.10}$$

where

$$\kappa^2 \equiv \frac{\lambda + 2\mu}{\mu} = \left(\frac{c_{\rm p}}{c_{\rm s}}\right)^2. \tag{A.11}$$

Using the relation

$$\nabla^2\phi + k_{\rm p}^2\phi = 0, \tag{A.12}$$

the first term of the last equation of Eq. (A.10) is described as follows:

$$\begin{aligned}\kappa^2\nabla^2\phi &= \left(\frac{c_{\mathrm{p}}}{c_{\mathrm{s}}}\right)^2\nabla^2\phi \\ &= \left(\frac{k_{\mathrm{s}}}{k_{\mathrm{p}}}\right)^2\nabla^2\phi \\ &= -k_{\mathrm{s}}^2\phi. \qquad \text{(A.13)}\end{aligned}$$

Thus, using Eqs. (A.1) and (A.2), the stress σ_{yy} of Eq. (A.10) is given by

$$\begin{aligned}\sigma_{yy} &= -\mu\left\{k_{\mathrm{s}}^2\phi + 2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\psi}{\partial x\partial y}\right)\right\} \\ &= -\mu\left\{(k_{\mathrm{s}}^2 - 2k_{\mathrm{L}}^2)A\sinh(\eta y) + (2jk_{\mathrm{L}}\beta)B\sinh(\beta y)\right\}\exp(jk_{\mathrm{L}}x). \qquad \text{(A.14)}\end{aligned}$$

On the other hand, using Eqs. (A.8) and (A.9), the y-direction shear stress in the plane normal to x-axis, σ_{xy}, is given by

$$\begin{aligned}\sigma_{xy} &\equiv 2\mu \times \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \text{ [Pa]} \\ &= \mu\left\{(2jk_{\mathrm{L}}\eta)A\cosh(\eta y) + (k_{\mathrm{L}}^2 + \beta^2)B\cosh(\beta y)\right\}\exp(jk_{\mathrm{L}}x). \qquad \text{(A.15)}\end{aligned}$$

Since the plate is immersed in blood, the acoustic energy of the Lamb wave in the plate leaks into the surrounding blood medium. For the leaky component, the potential ϕ_{b} of the primary wave in the $x - y$ plane in blood is given by

$$\phi_{\mathrm{b}} = \begin{cases} -C\exp(-\eta_{\mathrm{b}}y)\exp(jk_{\mathrm{b}}x) & \text{if } y > 0 \\ C\exp(\eta_{\mathrm{b}}y)\exp(jk_{\mathrm{b}}x) & \text{if } y < 0 \end{cases}, \qquad \text{(A.16)}$$

where k_{b} is the wave number in blood and C is an amplitude constant. The minus sign of the coefficient $(-C)$ of the first equation in Eq. (A.16) characterizes the asymmetric mode. Using the velocity c_{b} for the primary wave in blood, the following η_{b} and k_{b} are defined by

$$\eta_b = \sqrt{k_{\mathrm{L}}^2 - k_{\mathrm{b}}^2} \text{ [rad/m]} \qquad \text{(A.17)}$$

$$k_{\mathrm{b}} = \frac{\omega}{c_{\mathrm{b}}} \text{ [rad/m]}. \qquad \text{(A.18)}$$

For the primary wave in blood leaked from the IVS boundary, the displacement u_y in the y-direction, the displacement u_x in the x-direction, and stress σ_{yy} normal to the y-axis are respectively given by

$$
\begin{aligned}
u_y &\equiv \frac{\partial \phi_{\mathrm{b}}}{\partial y} \text{ [m]} \\
&= \begin{cases} \eta_{\mathrm{b}} C \exp(-\eta_{\mathrm{b}} y) \exp(jk_{\mathrm{b}} x) & \text{if } y > 0 \\ \eta_{\mathrm{b}} C \exp(\eta_{\mathrm{b}} y) \exp(jk_{\mathrm{b}} x) & \text{if } y < 0 \end{cases}, \qquad \text{(A.19)}
\end{aligned}
$$

$$
\begin{aligned}
u_x &\equiv \frac{\partial \phi_{\mathrm{b}}}{\partial x} \text{ [m]} \\
&= \begin{cases} -jk_{\mathrm{b}} C \exp(-\eta_{\mathrm{b}} y) \exp(jk_{\mathrm{b}} x) & \text{if } y > 0 \\ jk_{\mathrm{b}} C \exp(\eta_{\mathrm{b}} y) \exp(jk_{\mathrm{b}} x) & \text{if } y < 0 \end{cases}, \qquad \text{(A.20)}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{yy} &\equiv \lambda_{\mathrm{b}} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) \text{ [Pa]} \\
&= \lambda_{\mathrm{b}} \left(\frac{\partial^2 \phi_{\mathrm{b}}}{\partial x^2} + \frac{\partial^2 \phi_{\mathrm{b}}}{\partial y^2} \right) \\
&= \lambda_{\mathrm{b}} \nabla^2 \phi_{\mathrm{b}} \\
&= -\lambda_{\mathrm{b}} k_{\mathrm{b}}^2 \phi_{\mathrm{b}} \\
&= -\rho_{\mathrm{b}} \omega^2 \phi_{\mathrm{b}} \\
&= \begin{cases} \rho_{\mathrm{b}} \omega^2 C \exp(-\eta_{\mathrm{b}} y) \exp(jk_{\mathrm{b}} x) & \text{if } y > 0 \\ -\rho_{\mathrm{b}} \omega^2 C \exp(\eta_{\mathrm{b}} y) \exp(jk_{\mathrm{b}} x) & \text{if } y < 0 \end{cases}, \qquad \text{(A.21)}
\end{aligned}
$$

where λ_{b} is the Lamé constant in blood, ρ_{b} is the blood density ($= 1.1 \times 10^3$ kg/m^3), $\partial u_x / \partial x = -k_{\mathrm{b}}^2 \phi_{\mathrm{b}}$, and $\partial u_y / \partial y = \eta_{\mathrm{b}}^2 \phi_{\mathrm{b}}$, and the following relation is used.

$$
k_{\mathrm{b}} = \omega \sqrt{\frac{\rho_{\mathrm{b}}}{\lambda_{\mathrm{b}}}} \qquad \text{(A.22)}
$$

By applying the vanishing shear stress (σ_{xy} of Eq. (A.15)) condition at the myocardium-blood interfaces ($y = \pm h$), continuity of the normal stress (σ_{yy} of Eqs. (A.14) and (A.21)), and the continuity of the displacement (u_y of Eqs. (A.9) and (A.19)) across the two interfaces at $y = \pm h$, the following three equations are obtained.

$$
\begin{aligned}
\sigma_{xy}\Big|_{y=\pm h} &= \mu \left\{ (2jk_{\mathrm{L}} \eta) A \cosh(\eta h) + (k_{\mathrm{L}}^2 + \beta^2) B \cosh(\beta h) \right\} \exp(jk_{\mathrm{L}} x) \\
&= \mu \left\{ (2jk_{\mathrm{L}} \eta) A \cosh(\eta h) + (2k_{\mathrm{L}}^2 - k_{\mathrm{s}}^2) B \cosh(\beta h) \right\} \exp(jk_{\mathrm{L}} x) \\
&= 0 \qquad \text{(A.23)}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{yy}\Big|_{y=\pm h} &= -\mu \left\{ \pm (k_{\mathrm{s}}^2 - 2k_{\mathrm{L}}^2) A \sinh(\eta h) \pm (2jk_{\mathrm{L}} \beta) B \sinh(\beta h) \right\} \exp(jk_{\mathrm{L}} x) \\
&= \pm \rho_{\mathrm{b}} \omega^2 C \exp(-\eta_{\mathrm{b}} h) \exp(jk_{\mathrm{L}} x) \qquad \text{(A.24)}
\end{aligned}
$$

$$u_y\Big|_{y=\pm h} = \Big\{\eta A\cosh(\eta h) - jk_{\mathrm{L}}B\cosh(\beta h)\Big\}\exp(jk_{\mathrm{L}}x)$$
$$= \eta_{\mathrm{b}}C\exp(-\eta_{\mathrm{b}}h)\exp(jk_{\mathrm{L}}x). \quad \text{(A.25)}$$

Since these three equations hold for all x, the term $\exp(jk_{\mathrm{L}}x)$ in both sides can be eliminated and the remaining equations are rewritten in the matrix form

$$\begin{pmatrix} 2jk_{\mathrm{L}}\eta\cosh(\eta h) & (2k_{\mathrm{L}}^2 - k_{\mathrm{s}}^2)\cosh(\beta h) & 0 \\ (2k_{\mathrm{L}}^2 - k_{\mathrm{s}}^2)\sinh(\eta h) & -2jk_{\mathrm{L}}\beta\sinh(\beta h) & -\frac{\rho_{\mathrm{b}}k_{\mathrm{s}}^2}{\rho_{\mathrm{m}}}\exp(-\eta_{\mathrm{b}}h) \\ \eta\cosh(\eta h) & -jk_{\mathrm{L}}\cosh(\beta h) & -\eta_{\mathrm{b}}\exp(-\eta_{\mathrm{b}}h) \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{(A.26)}$$

where the myocardial density ρ_{m} is introduced based on the relationship of $k_{\mathrm{s}} = \omega\sqrt{\rho_{\mathrm{m}}/\mu}$ of Eq. (A.7). For nontrivial solution of A, B, and C, the following determinant Δ of the 3×3 matrix should be zero.

$$\Delta = -\eta_{\mathrm{b}}\exp(-\eta_{\mathrm{b}}h)\Big\{4k_{\mathrm{L}}^2\eta\beta\cosh(\eta h)\sinh(\beta h) - (2k_{\mathrm{L}}^2 - k_{\mathrm{s}}^2)^2\sinh(\eta h)\cosh(\beta h)\Big\}$$
$$+\frac{\rho_{\mathrm{b}}k_{\mathrm{s}}^2}{\rho_{\mathrm{m}}}\exp(-\eta_{\mathrm{b}}h)\Big\{2k_{\mathrm{L}}^2\eta - (2k_{\mathrm{L}}^2 - k_{\mathrm{s}}^2)\eta\Big\}\cosh(\eta h)\cosh(\beta h)$$
$$= 0. \quad \text{(A.27)}$$

Therefore, the following function, termed $f(k_{\mathrm{L}}, k_{\mathrm{p}}, k_{\mathrm{s}})$, should zero.

$$f(k_{\mathrm{L}}, k_{\mathrm{p}}, k_{\mathrm{s}}) \equiv 4k_{\mathrm{L}}^2\eta\beta\cosh(\eta h)\sinh(\beta h) - (2k_{\mathrm{L}}^2 - k_{\mathrm{s}}^2)^2\sinh(\eta h)\cosh(\beta h)$$
$$-\frac{\rho_{\mathrm{b}}\eta k_{\mathrm{s}}^4}{\rho_{\mathrm{m}}\eta_{\mathrm{b}}}\cosh(\eta h)\cosh(\beta h) = 0. \quad \text{(A.28)}$$

This function is employed in Eq. (2) of Section III-B.