Accurate Autoregressive Spectrum Estimation at Low Signal-to-Noise Ratio Using a Phase Matching Technique

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Abstract—**This paper describes a new method of accurately estimating the parameters of an autoregressive (AR) process contaminated by high-level white noise. Based on the phase matching technique, it minimizes the difference between the phase of the all-zero model and the phase of the maximum phase signal reconstructed from the power spectrum of the observed signal. The parameters of the AR model are obtained from the finite length sequence of the estimated all-zero model. The proposed method works only when the order of the AR model is known *a priori* at present. However, since the phase matching technique satisfies the conditions needed to apply the least mean-square method, the AR parameters are estimated accurately even at a low signal-to-noise ratio. With the iterative or noniterative methods as discussed in the recent literature, it is not possible to reconstruct the all-zero model from the power spectrum when there are dips and peaks having no correlation with the poles of original AR signal in the power spectrum. The method proposed in this paper allows one to accurately reconstruct the phase from the power spectrum in such cases. Finally, it is confirmed with computer simulations and experiments that the proposed method is useful for accurate estimation of the AR parameters.**

I. Introduction

THIS paper concerns a method for describing the resonant vibration system using the parameters of an AR model. When noise is added to a signal under analysis, the signal is not described by the AR process, even if the signal is the response of an AR system because of the introduction of the spectral zeros by the additive noise. A number of methods for estimating AR parameters of such AR-plus-noise models have been proposed recently in the literature [1]–[5]. However, accurate AR parameters of the model are not obtained by such methods as described in Section II.

Therefore, we have developed a new method of estimating AR parameters accurately, even when the signal is contaminated by white noise. The proposed method is based on a significant extension of signal reconstruction techniques using the phase estimated from the observed signal. Since the error in the estimated phase is distributed according to the normal distribution with constant mean and variance for all frequencies as described in Section III, accurate AR parameters are obtained by a least square fit with the phase error even in the case of low signal-to-noise ratio (SNR). The proposed method involves nonlinear optimization. However, its computational requirements are quite modest as shown by the simulation experiments in Section V. Finally, in Section VI, other advantages of the proposed method are also shown by experiments involving resonant vibration generated in a mechanical system.

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II. Disadvantages of Recent Methods for Estimating AR Parameters

If a signal $x(n)$ is contaminated by white noise $w(n)$, the observed signal $y(n)$ is described as follows:

$$y(n) = x(n) + w(n). \tag{1}$$

If $x(n)$ is the output signal of a pth-order AR model excited by white noise $e(n)$,

$$x(n) = -\sum_{i=1}^{p} a_i \cdot x(n-i) + e(n) \tag{2}$$

where $e(n)$ is the white noise uncorrelated with $w(n)$ and

$$\text{var}\left\{e(n)\right\} = \sigma_e^2$$
$$\text{var}\left\{w(n)\right\} = \sigma_w^2.$$

Each z transform of (1) and (2) is represented as follows:

$$Y(z) = X(z) + W(z) \tag{1'}$$

and

$$X(z) = \frac{1}{A(z)} \cdot E(z) \tag{2'}$$

where $A(z) = 1 + a_1 z^{-1} + \cdots + a_p z^{-p}$, and $Y(z)$, $X(z)$, $W(z)$, and $E(z)$ denote the z transforms of $y(n)$, $x(n)$, $w(n)$, and $e(n)$, respectively. The power spectrum $P_y(z)$ of the observed signal $y(n)$ is expressed as follows:

$$P_y(z) = \frac{\sigma_e^2}{A(z) \cdot A^*(1/z^*)} + \sigma_w^2 = \frac{\sigma_e^2 + \sigma_w^2 \cdot A(z) \cdot A^*(1/z^*)}{A(z) \cdot A^*(1/z^*)} \tag{3}$$

where z^* denotes the complex conjugate of z. Let the numerator of the last equation be $\sigma_u^2 \cdot B(z) \cdot B^*(1/z^*)$ where $B(z) = 1 + \Sigma_{i=1}^{p} b_i \cdot z^{-i}$; then $y(n)$ may be modeled as the ARMA(p, p) model $B(z)/A(z)$ driven by the white noise $u(n)$ of the power σ_u^2. The z transform of the output signal $y(n)$ is expressed as follows:

$$Y(z) = U(z)\frac{B(z)}{A(z)} \tag{4}$$

where $U(z)$ is the z transform of $u(n)$. Thus, the observation noise introduces spectral zeros, which are located between the true AR poles and the origin in the z plane. The zeros move toward the true AR poles as the SNR decreases [3].

A number of methods for estimating AR parameters in such cases have been proposed in the recent literature [1]–[5] and can be divided into the following five classes [1], [2].

1) The larger order AR spectral estimation in which the model is approximated to a larger order AR model [3].

2) The method using modified Yule–Walker equations.

3) The noise compensation method for the correlation function in which the estimated additive noise power is subtracted from the main diagonal of the autocorrelation matrix [4].

4) The direct power spectrum matching technique which minimizes the average squared difference between the nonparametric power spectrum (= periodogram) estimated from the observed signal and the parametric power spectrum of the model [5].

5) The method using the overdetermined modified Yule–Walker equations [2].

All of these five methods estimate the AR parameters using the autocorrelation function estimate $\boldsymbol{R}(\tau)$ or its equivalent, the power spectrum estimate $\boldsymbol{P}_y(\omega)$, which is equal to the periodogram in the fourth method 4) where boldface is used throughout the paper to distinguish estimates from the true values. However, accurate AR parameters cannot be obtained by those methods as described below. Each of the real and imaginary parts of the spectrum $\boldsymbol{Y}(\omega)$, estimated from the AR-plus-noise signal $y(n)$ of (4), is distributed independently according to the normal distribution $N[0, \sigma_v^2(\omega)]$ where the variance is $\sigma_v^2(\omega) = |B(\omega)/A(\omega)|^2 \cdot \sigma_u^2 = P_y(\omega)$ and $P_y(\omega)$ denotes the true power spectrum of $y(n)$. Therefore, the periodogram estimate $\boldsymbol{P}_y(\omega)$ of $y(n)$ is distributed according to the chi-squared distribution with 2 degrees of freedom where the mean and the variance are $\sigma_v^2(\omega) = P_y(\omega)$ and $\sigma_v^4(\omega) = P_y^2(\omega)$, respectively [6, ch. 11.3]. Thus, the periodogram $\boldsymbol{P}_y(\omega)$ has a large variance, especially in the frequency band where the signal-to-noise ratio is high. Therefore, accurate AR parameters are not obtained by the methods described above, even if the (generalized) least square fitting technique is applied to the minimization of the average squared errors involved in the estimated power spectrum or the estimated autocorrelation.

III. Phase Matching Based Method for Estimating AR Parameters

Here, we propose a new method for estimating the AR parameters. The following discussion is based on the assumption that the real signal $y(n)$ has a rational z transform of the AR-plus-noise signal. The five processes of the method are explained below using the simple example of a first-order AR model which has a pole z_a and the introduced spectral zero z_b, ($|z_b| < |z_a| < 1$) in the z plane. By using the same example, the right-hand side of Fig. 1 shows schematically the processes of the method, and the left-hand side of Fig. 1 illustrates the processes by applying them to the second-order AR model used in Section V.

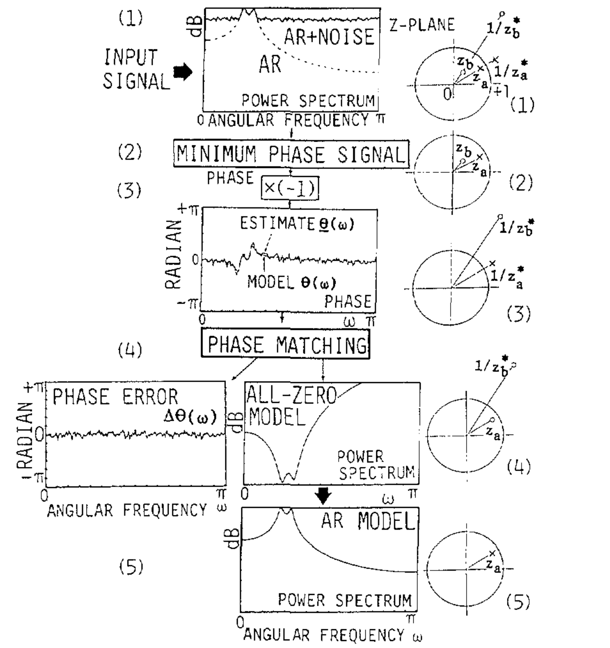


Fig. 1. Illustration showing the procedure of the proposed method for estimation of the parameters of the AR process in white noise.

1) The power spectrum estimate $\boldsymbol{P}_y(\omega)$, which is the averaged periodogram, is obtained using FFT of the observed signal $y(n)$ [see Fig. 1, part (1)]. Since $\boldsymbol{P}_y(\omega)$ has the pole z_a, the zero z_b, and their complex conjugate reciprocal pairs $1/z_a^*$ and $1/z_b^*$, the power spectrum $P_y(z)$ is expressed as follows:

$$\begin{aligned} P_y(z) &= \sigma_u^2 \cdot \frac{B(z) \cdot B^*(1/z^*)}{A(z) \cdot A^*(1/z^*)} \\ &= \sigma_u^2 \cdot \frac{(1 - z_b z^{-1})(1 - z_b^* z)}{(1 - z_a z^{-1})(1 - z_a^* z)}. \end{aligned} \tag{5}$$

2) Based on the noniterative minimum phase reconstruction algorithm [7], [8], the minimum phase signal $X_{\min}(z)$ is reconstructed from the periodogram $\boldsymbol{P}_y(\omega)$ using FFT two times. Since this operation is equivalent to mapping maximum phase poles and zeros to their conjugate symmetric counterparts with the unit circle of the z plane, $X_{\min}(z)$ is obtained as follows:

$$X_{\min}(z) = C_0 \cdot \frac{B(z)}{A(z)} = C_0 \cdot \frac{(1 - z_b z^{-1})}{(1 - z_a z^{-1})} \tag{6}$$

where $C_0 = \sqrt{\sigma_u^2}$ [see Fig. 1, part (2)]. This equation shows that the minimum phase reconstructed signal

$X_{\min}(z)$ is the output of the ARMA(p, p) model $B(z)/A(z)$ excited by the impulse $C_0 \cdot \delta(n)$.

3) The phase $\Theta_{\min}(\omega)$ of the minimum phase signal $X_{\min}(z)$ is calculated. By changing the sign of $\Theta_{\min}(\omega)$, the new phase function $\Theta(\omega)$ is obtained. Since a) by mapping the poles and the zeros to their conjugate symmetric counterparts with the unit circle of the z plane, the sign of the phase function is changed, and b) the pole $1/z_a^*$ or the zero $1/z_b^*$ makes the same contribution to the phase as the zero z_a or the pole z_b, respectively, the following four models have the same phases as $\Theta(\omega)$:

$$X'_{\min}(z) = C_0 \cdot \frac{(1 - z_a z^{-1})}{(1 - z_b z^{-1})},$$

$$X_{\text{pole}}(z) = C_0 \cdot \frac{1}{(1 - z_a^* z)(1 - z_b z^{-1})},$$

$$X_{\max}(z) = C_0 \cdot \frac{(1 - z_b^* z)}{(1 - z_a^* z)},$$

and

$$X_{\text{zero}}(z) = C_0 \cdot (1 - z_a z^{-1})(1 - z_b^* z). \tag{7}$$

4) By restricting the signal reconstructed from the phase $\Theta(\omega)$ to being finite length, only the all-zero model $X_{\text{zero}}(z)$, having the same phase as $\Theta(\omega)$, is obtained [see Fig. 1, part (4)]. A method of obtaining the signal from its phase is described in Section IV.

5) As the method ordinarily used in the linear prediction of speech to determine the poles of the AR model [9, p. 29], the poles of the AR model are determined from the roots of the polynomial:

$$X_{\text{zero}}(z) = \sum_{n=-p}^{p} x_{\text{zero}}(n) \cdot z^{-n} = 0. \tag{8}$$

Since $x_{\text{zero}}(-p)$ is not zero, by dividing (8) by $x_{\text{zero}}(-p) \cdot z^p$, the following polynomial is obtained:

$$\sum_{n=0}^{2p+1} x'_{\text{zero}}(n) \cdot z^{-n} = 0 \tag{9}$$

where $x'_{\text{zero}}(n) = x_{\text{zero}}(n - p)/x_{\text{zero}}(-p)$. The roots of the polynomial contain the pole z_a ($|z_a| < 1$) and the zero $1/z_b^*$ ($|1/z_b^*| > 1$) in (5). If the roots whose magnitudes are less than one are selected, only the poles of the original AR model are obtained [see Fig. 1, part (5)].

As described in Section II, almost all the methods in the literature have estimated AR parameters using the estimated power spectrum (= periodogram) or its equivalent, the estimated autocorrelation function. However, the variance of the errors in the periodogram estimate is equal to $P_y^2(\omega)$ and is large, especially in the frequency band where the poles make peaks. However, the variance of the error $\Delta\Theta(\omega)$ of the phase $\Theta(\omega)$ estimated *in the third* process of the proposed method is constant for all the frequencies as described in the following.

Since the observed signal is the finite length, there are many peaks and dips having no correlation with the poles of the original AR model in the estimated power spectrum $\boldsymbol{P}_y(\omega)$. Thus, in a practical case, the minimum phase signal $\boldsymbol{X}_{\min}(z)$ of (6) calculated from $\boldsymbol{P}_y(\omega)$ is expressed as the sum of two components:

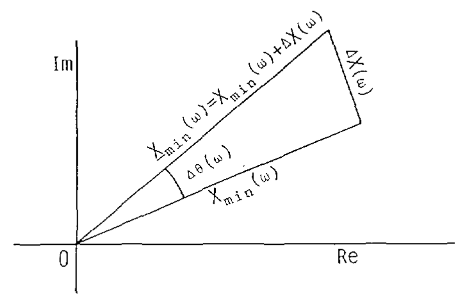


Fig. 2. Illustration showing the phase error $\Delta\Theta(\omega)$ in the minimum phase model $\boldsymbol{X}_{\min}(\omega)$ which is reconstructed from the power spectrum of the observed signal $y(n)$.

$$\boldsymbol{X}_{\min}(z) = C_0 \cdot \frac{B(z)}{A(z)} \cdot (1 + \Delta U(z)) \tag{10}$$

where $\Delta U(z)$ denotes approximately the z transform of Gauss random noise $\Delta u(n)$. Then, the spectrum $\boldsymbol{X}_{\min}(\omega)$ of the estimated minimum phase model is expressed as follows:

$$\boldsymbol{X}_{\min}(\omega) = X_{\min}(\omega) + \Delta X(\omega) \tag{11}$$

where $X_{\min}(\omega)$ and $\Delta X(\omega)$ denote the spectrum of the true minimum phase model and the spectrum error, respectively, as shown in Fig. 2. Using (10), $X_{\min}(\omega)$ and $\Delta X(\omega)$ are expressed as follows:

$$X_{\min}(\omega) = C_0 \cdot \frac{B(\omega)}{A(\omega)} \tag{12a}$$

and

$$\begin{aligned}\Delta X(\omega) &= C_0 \cdot \frac{B(\omega)}{A(\omega)} \cdot \Delta U(\omega) \\ &= X_{\min}(\omega) \cdot \Delta U(\omega).\end{aligned} \tag{12b}$$

Thus, the estimated phase $\boldsymbol{\Theta}_{\min}(\omega)$ of the minimum phase signal $\boldsymbol{X}_{\min}(z)$ is expressed as the sum of the true phase $\Theta_{\min}(\omega)$ and the phase error $\Delta\Theta(\omega)$:

$$\boldsymbol{\Theta}_{\min}(\omega) = \Theta_{\min}(\omega) + \Delta\Theta(\omega). \tag{13}$$

The left-hand side of Fig. 1, part (3) shows that true phase $\Theta(\omega) = -\Theta_{\min}(\omega)$ and its estimate $\boldsymbol{\Theta}(\omega) = -\boldsymbol{\Theta}_{\min}(\omega)$. When $|\Delta\Theta(\omega)|$ is significantly less than 1.0, $\Delta\Theta(\omega)$ is approximated by $\tan \Delta\Theta(\omega)$. Then, $\Delta\Theta(\omega)$ is expressed as follows:

$$\begin{aligned}\Delta\Theta(\omega) &\simeq \tan \Delta\Theta(\omega) \\ &= \tan\left\{\boldsymbol{\Theta}_{\min}(\omega) - \Theta_{\min}(\omega)\right\} \\ &= \frac{\tan \boldsymbol{\Theta}_{\min}(\omega) - \tan \Theta_{\min}(\omega)}{1 + \tan \boldsymbol{\Theta}_{\min}(\omega) \cdot \tan \Theta_{\min}(\omega)}.\end{aligned} \tag{14}$$

Each spectrum of (11) is decomposed into real and imaginary parts as follows:

$$X_{\min}(\omega) = R_{\min}(\omega) + jI_{\min}(\omega), \quad (15a)$$

$$\Delta X(\omega) = \Delta R(\omega) + j\Delta I(\omega), \quad (15b)$$

and

$$X_{\min}(\omega) = [R_{\min}(\omega) + \Delta R(\omega)] + j[I_{\min}(\omega) + \Delta I(\omega)]. \quad (15c)$$

Using these spectra, the phase error (14) is expressed as follows:

$$\Delta\Theta(\omega) = \frac{\{I_{\min}(\omega) + \Delta I(\omega)\} R_{\min}(\omega) - \{R_{\min}(\omega) + \Delta R(\omega)\} I_{\min}(\omega)}{\{R_{\min}(\omega) + \Delta R(\omega)\} R_{\min}(\omega) + \{I_{\min}(\omega) + \Delta I(\omega)\} I_{\min}(\omega)}. \quad (16)$$

If $|\Delta R(\omega)| << |R_{\min}(\omega)|$ and $|\Delta I(\omega)| << |I_{\min}(\omega)|$, the phase error is approximated by the following equation:

$$\begin{aligned}\Delta\Theta(\omega) &\simeq \frac{\Delta I(\omega) R_{\min}(\omega) - \Delta R(\omega) I_{\min}(\omega)}{R_{\min}(\omega)^2 + I_{\min}(\omega)^2} \\ &= \frac{\mathrm{Im}\,[\Delta X(\omega) \cdot X^*_{\min}(\omega)]}{|X_{\min}(\omega)|^2} \\ &= \mathrm{Im}\,[\Delta X(\omega)/X_{\min}(\omega)] \\ &= \mathrm{Im}\,[\Delta U(\omega)]. \end{aligned} \quad (17)$$

That is, the phase error is equal to the imaginary part of the spectrum $\Delta U(\omega)$ of the Gauss random noise in (10). Thus, the phase errors $\{\Delta\Theta(\omega)\}$ shown in Fig. 1, part (4) are distributed according to the normal distribution $N[0, \sigma^2]$ where the variance is constant for all the frequencies. Therefore, more accurate AR parameters are obtained by the least mean-square fit of the phase of the all-zero model $X_{\text{zero}}(z)$ to the phase $\mathbf{\Theta}(\omega) = -\mathbf{\Theta}_{\min}(\omega)$ estimated from the observed signal. A phase matching method is proposed in Section IV-C.

IV. Reconstruction of an All-Zero Model from Its Phase

In the literature, there are two methods for reconstructing a finite duration mixed phase signal from its phase. In this section, the difficulties in reconstructing the signal from the noisy phase $\mathbf{\Theta}(\omega)$ using these methods are explained, and then an alternative new reconstruction method is proposed.

A. Noniterative Method [7], [10]–[11]

This method reconstructs a finite length $(= 2p + 1)$ signal $x_{\text{zero}}(n)$, $(-p \le n \le +p)$ of the all-zero model $X_{\text{zero}}(z)$ from the phase estimate $\mathbf{\Theta}(\omega)$. Since the signal $x_{\text{zero}}(n)$ of the all-zero model is real, the following relation holds well for the discrete Fourier transform $X_{\text{zero}}(\omega)$:

$$\frac{X_{\text{zero}}(+\omega)}{X_{\text{zero}}(-\omega)} = \frac{\exp\{+j\Theta(\omega)\}}{\exp\{-j\Theta(\omega)\}} \quad (\text{for all } \omega). \quad (18)$$

Therefore,

$$X_{\text{zero}}(+\omega) \exp\{-j\Theta(\omega)\} = X_{\text{zero}}(-\omega) \exp\{+j\Theta(\omega)\}. \quad (19)$$

Using the definition of the Fourier transform, the imaginary part of the equation is expressed as follows:

$$\sum_{n=-p}^{p} x_{\text{zero}}(n) \cdot \sin\{n\omega + \Theta(\omega)\} = 0 \quad (\text{for all } \omega). \quad (20)$$

Since $\Theta(\omega)$ is selected from $(2p + 1)$ distinct values of ω, $(2p + 1)$ linear homogeneous equations are obtained from (20).

When $\Theta(\omega)$ expresses the true phase of the all-zero model, $x_{\text{zero}}(n)$ can be determined from the linear equations. However, there is a phase error of $\Delta\Theta(\omega)$ in phase $\Theta(\omega)$ calculated from the estimated power spectrum as shown in Fig. 1, part (3), and this method does not compensate for the phase error at all. Therefore, an accurate all-zero model and AR parameters cannot be obtained using this method.

B. Iterative Method [7], [10]–[11]

Let $x_{\text{zero}}(n:i)$ be the estimate of the all-zero model at the ith iteration. In the frequency domain, the phase of $x_{\text{zero}}(n:i)$ is replaced by the estimated phase $\mathbf{\Theta}(\omega)$, and in the time domain, the finite duration constraint is imposed. Then, the $(i+1)$th estimate is obtained, and the unknown spectrum magnitude is retrieved gradually in successive iterations.

When $\mathbf{\Theta}(\omega)$ can accurately express the phase of the all-zero model, it is known that the iteration procedure always converges to a unique limit of $A \cdot x_{\text{zero}}(n)$ where A is a positive constant [10]–[11]. However, as shown in Fig. 1, part (3), there is a difference of $\{\Delta\Theta(\omega)\}$ between the obtained phase $\mathbf{\Theta}(\omega)$ and the true phase $\Theta(\omega)$ of the all-zero model as follows:

$$\Delta\Theta(\omega) = \mathbf{\Theta}(\omega) - \Theta(\omega) \quad (21)$$

where $\Delta\Theta(\omega)$ of this equation is equal to $-\Delta\Theta(\omega)$ of (13). Thus, the satisfying all-zero model is not reconstructed. This may be explained as follows. Let $x_{\text{zero}}(n)$ and $\mathbf{x}_{\text{zero}}(n)$ be the signals of all-zero models reconstructed from the true phase $\Theta(\omega)$ and its estimate $\mathbf{\Theta}(\omega)$, respectively. Substituting $\mathbf{\Theta}(\omega)$ of (21) into (19), the spectrum $X_{\text{zero}}(\omega)$ of $x_{\text{zero}}(n)$ is expressed as follows:

$$\exp[-j\{\mathbf{\Theta}(\omega) - \Delta\Theta(\omega)\}] \cdot X_{\text{zero}}(\omega) = \exp[j\{\mathbf{\Theta}(\omega) - \Delta\Theta(\omega)\}] \cdot X_{\text{zero}}(-\omega). \quad (22)$$

Comparing (22) to (19), the spectrum $\mathbf{X}_{\text{zero}}(\omega)$ of $\mathbf{x}_{\text{zero}}(n)$

is expressed by [10, sect. VI]

$$\boldsymbol{X}_{\text{zero}}(\omega) = \frac{1}{\lambda} \cdot \exp\left\{j\Delta\Theta(\omega)\right\} \cdot X_{\text{zero}}(\omega) \quad (23)$$

where λ is the constant ensuring that $\boldsymbol{x}_{\text{zero}}(0) = 1$; assuming that $|\Delta\Theta(\omega)| << 1$, $\boldsymbol{X}_{\text{zero}}(\omega)$ of (23) is approximated by

$$\boldsymbol{X}_{\text{zero}}(\omega) \simeq \frac{1}{\lambda} \cdot \left\{1 + j\Delta\Theta(\omega)\right\} \cdot X_{\text{zero}}(\omega). \quad (24)$$

Thus, $\boldsymbol{x}_{\text{zero}}(n)$ is described by

$$\boldsymbol{x}_{\text{zero}}(n) = \frac{1}{\lambda} \cdot \left\{x_{\text{zero}}(n) + x_{\text{zero}}(n)^* \, e(n)\right\} \quad (25)$$

where $e(n)$ is the real odd signal of the inverse Fourier transform of the imaginary odd sequence $j\Delta\Theta(\omega)$. Since the phase errors $\{\Delta\Theta(\omega)\}$ are distributed randomly according to the normal distribution as described in Section III, $e(n)$ has an infinite duration, and consequently, the duration of the reconstructed signal $\boldsymbol{x}_{\text{zero}}(n)$ is also infinite. However, the finite duration constraint is imposed in each iteration. Therefore, an accurate all-zero model cannot be obtained by this method.

C. A Proposed Reconstruction Method Using the Least Square Method Which Fits the Phase Error

Substitution of $\Theta(\omega)$ of (21) into (20) results in

$$\sum_{n=-p}^{p} x_{\text{zero}}(n) \cdot \sin\left\{n\omega + \boldsymbol{\Theta}(\omega) - \Delta\Theta(\omega)\right\} = 0 \quad \text{(for all } \omega). \quad (26)$$

Using the formula $\sin(\Omega_1 - \Omega_2) = \sin\Omega_1 \cdot \cos\Omega_2 - \cos\Omega_1 \cdot \sin\Omega_2$, and assuming that $|\Delta\Theta(\omega)| << 1$, this equation is approximated by

$$\sum_{n=-p}^{p} x_{\text{zero}}(n) \cdot \left[\sin\left\{n\omega + \boldsymbol{\Theta}(\omega)\right\} \cdot \cos\left\{n\omega + \boldsymbol{\Theta}(\omega)\right\}\right] \simeq 0 \quad \text{(for all } \omega). \quad (27)$$

Then, the phase error $\Delta\Theta(\omega)$ is expressed as follows:

$$\Delta\Theta(\omega) = \frac{\sum_{n=-p}^{p} x_{\text{zero}}(n) \cdot \sin\left\{n\omega + \boldsymbol{\Theta}(\omega)\right\}}{\sum_{n=-p}^{p} x_{\text{zero}}(n) \cdot \cos\left\{n\omega + \boldsymbol{\Theta}(\omega)\right\}} \quad \text{(for all } \omega) \quad (28a)$$

where $x_{\text{zero}}(0) = 1$. When $\boldsymbol{\Theta}(\omega)$ is selected from equally distributed distinct M values with intervals of $2\pi/M$, the discrete phase error is expressed as follows:

$$\Delta\Theta(k) = \frac{\sum_{n=-p}^{p} x_{\text{zero}}(n) \cdot \sin\left\{2\pi(nk/M) + \boldsymbol{\Theta}(k)\right\}}{\sum_{n=-p}^{p} x_{\text{zero}}(n) \cdot \cos\left\{2\pi(nk/M) + \boldsymbol{\Theta}(k)\right\}} = \frac{\text{Im}\left[X_{\text{zero}}(-k) \cdot \exp\left\{j\boldsymbol{\Theta}(k)\right\}\right]}{\text{Re}\left[X_{\text{zero}}(-k) \cdot \exp\left\{j\boldsymbol{\Theta}(k)\right\}\right]} \quad \text{(for } k = 0, 1, 2, \cdots, M-1). \quad (28b)$$

The finite duration signal $x_{\text{zero}}(n)$ is obtained from the given phase $\boldsymbol{\Theta}(k)$ by minimizing the following sum α:

$$\alpha = \sum_{k=0}^{M-1} \left|\Delta\Theta(k)\right|^2 \rightarrow \text{MIN}. \quad (29)$$

This nonlinear minimization is performed using a standard optimization technique such as the Marquardt method [12]. Since the spectral zeros introduced by the observed noise are close to the poles of the AR model in the z plane, the z transform of the initial estimate $x_{\text{zero}}(n:0)$ of the iteration is selected as follows:

$$X_{\text{zero}}(z:0) = \prod_{i=1}^{p} (1 - z_i z^{-1})(1 - z_i^* z), \quad (30)$$

where $\{z_i\}$ are the poles obtained by the method using the overdetermined modified Yule–Walker equations.

If the denominator of the right-hand side in (28) is equal to zero, $\Delta\Theta(\omega)$ is evaluated as being unstable. Thus, the $(i+1)$th estimate $x_{\text{zero}}(n:i+1)$ is determined from the ith estimate $x_{\text{zero}}(n:i)$ under the condition that the denominator of the right-hand side in (28) can only be positive.

When the SNR is low, $|\Delta\Theta(\omega)|$ is less than $\pi/2$ for all ω. Thus, the initial estimate of the all-zero model $X_{\text{zero}}(z)$ satisfies the condition because every zero involved in $X_{\text{zero}}(z:0)$ of (30) has its complex reciprocal pair, and then the phase of $X_{\text{zero}}(z:0)$ is zero for all ω. The optimal estimate of the all-zero model also satisfies the condition because the phase of the denominator of (28) is described by $\{\text{phase of } X_{\text{zero}}(-\omega)\} + \boldsymbol{\Theta}(\omega) = \Delta\Theta(\omega)$, and $|\Delta\Theta(\omega)|$ is less than $\pi/2$ for all ω.

However, when the SNR is high, $|\Theta(\omega)|$ is not always less than $\pi/2$. In such cases, the positive constant β is added to the power spectrum estimate $\boldsymbol{P}_y(z)$ in (3) or (5) so that the absolute value of $\boldsymbol{\Theta}(\omega)$ calculated from the modified power spectrum $\boldsymbol{P}_y'(\omega) = \boldsymbol{P}_y(z) + \beta$ is less than $\pi/2$ for all ω. By adding β, the initial estimate satisfies the above condition, and each Marquardt iteration determines the successive estimate of the all-zero model under the above condition; the optimum estimate also satisfies the above condition. Since the variance of the phase errors $\Delta\Theta(\omega)$ does not increase by the addition of β, and the positions of all the poles involved in $P_y'(z)$ are equal to those of the poles in $\boldsymbol{P}_y(z)$; the AR parameters are obtained after convergence in the Marquardt method.

Since the phase reconstruction is nonlinear, the positive constant β is obtained in the following way. The calculation is in two stages, the first of which determines the minimum value β_0 of the real part of $\boldsymbol{X}_{\min}(\omega)$ reconstructed from $\boldsymbol{P}_y(\omega)$ for all ω. If β_0 is positive, $|\Theta(\omega)| = |-\Theta_{\min}(\omega)|$ is less than $\pi/2$ for all ω and both the all-zero model and AR parameters are estimated by minimizing α of (29) using $\Theta(\omega)$ without adding β_0 to $\boldsymbol{P}_y(\omega)$. If β_0 is not positive, in the second stage, $X_{\min}(\omega)$ is reconstructed from $\boldsymbol{P}_y'(\omega) = \boldsymbol{P}_y(\omega) + \beta$ where $\beta = \beta_0, 2\beta_0, 4\beta_0, 8\beta_0, \cdots, \beta_0'$ where β_0' is the first of these values at which the real part of $\boldsymbol{X}_{\min}(\omega)$ reconstructed from $\boldsymbol{P}_y'(\omega) = \boldsymbol{P}_y(\omega) + \beta_0'$ is positive for all ω. It then follows that

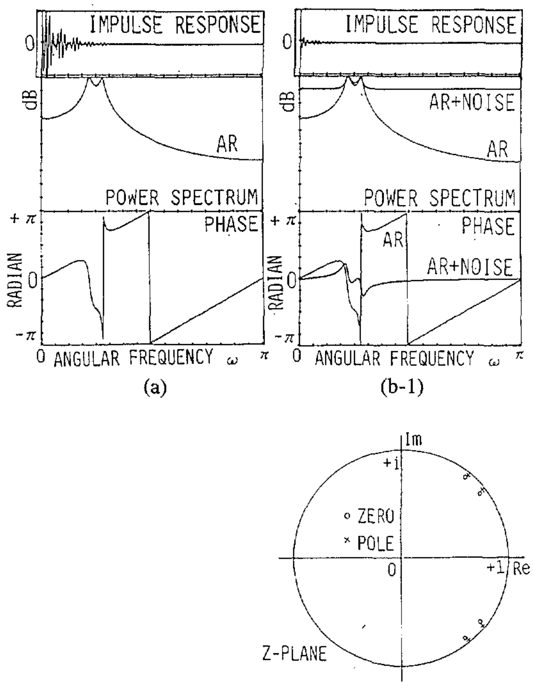


Fig. 3. The characteristics of the AR model (for the first example). (a) The fourth-order AR model of (31). (b-1) AR model buried in the white noise (SNR = -5 dB). (b-2) The poles of the AR model and the spectral zeros introduced by the additive noise.

the all-zero model and AR parameters can be estimated using $\Theta(\omega)$ obtained from $X_{\min}(\omega)$.

V. Simulation Results

In order to illustrate the characteristics of the proposed method and to compare the proposed method to the over-determined modified Yule–Walker (OMYM) approach [2, eq. (2.27)], we choose the most popular two examples used in the literature [4], [13].

First Example: Consider the following four-order AR process:

$$A(z) = \sum_{i=0}^{4} a_i \cdot z^{-i} \tag{31a}$$

where

$$a_0 = 1, \quad a_1 = -2.7607, \quad a_2 = 3.8106,$$
$$a_3 = -2.6535, \quad \text{and} \quad a_4 = 0.9238 \tag{31b}$$

Fig. 3(a) and (b) shows the characteristics of the AR model and the AR-plus-noise model (SNR = -5 dB), respectively. This AR process has four poles at both 0.98 exp ($\pm j0.69$) and 0.98 exp ($\pm j0.88$), which are close to each other as shown in Fig. 3(b-2). The two methods are implemented on a MELCOM-COSMO 700 (1 word ≒ 32 bit) computer using single precision arithmetic. The total length of the synthesized AR-plus-noise signal is 8192 points. The nonparametric power spectrum used in the proposed method is estimated by summation of 16 periodograms, each of which is computed using a Hanning window with a length of 512 points. In the OMYW method, by using the same AR-plus-noise process of 8191-point length, the extended Yule–Walker equations [2, eq. (2.23)] are evaluated for 20 distinct values of τ satisfying $\tau > 4$. Estimated AR parameters $\{\boldsymbol{a}_i\}$ obtained by the methods are evaluated by the following normalized mean-square error (NMSE) [14] as

$$\text{NMSE} = \frac{\sum_i (a_i - k \cdot \boldsymbol{a}_i)^2}{\sum_i a_i^2} \tag{32}$$

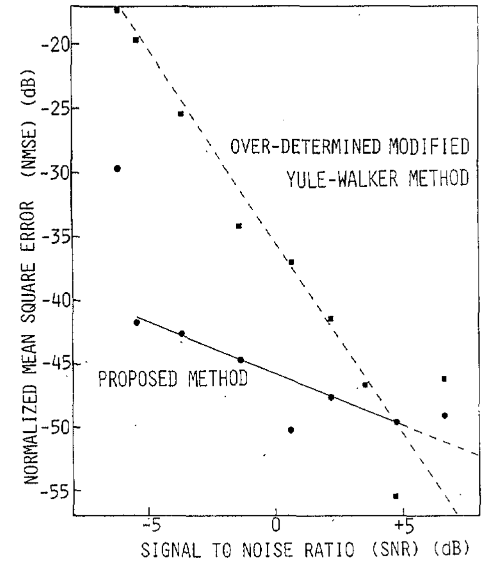


Fig. 4. The comparison of the AR parameters estimated by the two methods for various SNR (for the first example).

where k is the scaling constant chosen to minimize the NMSE. Typically, by using the proposed method, about 10–50 Marquardt iterations were needed to achieve convergence in the case considered here. Fig. 4 shows the NMSE as a function of the SNR. When the SNR is about +5 dB, both methods estimate the AR parameters accurately. In this case, the evaluation of the phase error $\Delta\Theta(\omega)$ of (28) is unstable since $|\Theta_{\min}(\omega)|$ of the minimum phase signal obtained from the power spectrum $P_y(\omega)$ has a value greater than $\pi/2$ for the frequencies around the central frequencies of the poles as shown Fig. 5(a), and the denominator of the right-hand side of (28) has small values for the frequencies. Then, as described previously in Section IV, by adding the positive constant β to the power spectrum $P_y(\omega)$ so that $|\Theta_{\min}(\omega)|$ calculated from $P_y'(\omega) = P_y(\omega) + \beta$ is less than $\pi/2$ for all ω as shown in Fig. 5(b), a satisfactory AR model is estimated by the proposed method as shown in Fig. 5(c).

When the additive noise level was increased by 5 dB, the error in estimated parameters obtained by the OMYW method increased by about 15 dB as shown in Fig. 4. Nevertheless, the proposed method estimates the AR parameters accurately, even in the lower SNR cases when the SNR ~ -5 dB as shown in Fig. 6.

Second Example: Consider the fourth-order AR process with parameters which are given as follows:

$$a_0 = 1, \quad a_1 = -1.352, \quad a_2 = +1.338,$$
$$a_3 = -0.662, \quad \text{and} \quad a_4 = +0.240. \tag{33}$$

Fig. 7 shows the characteristics of the AR process and the

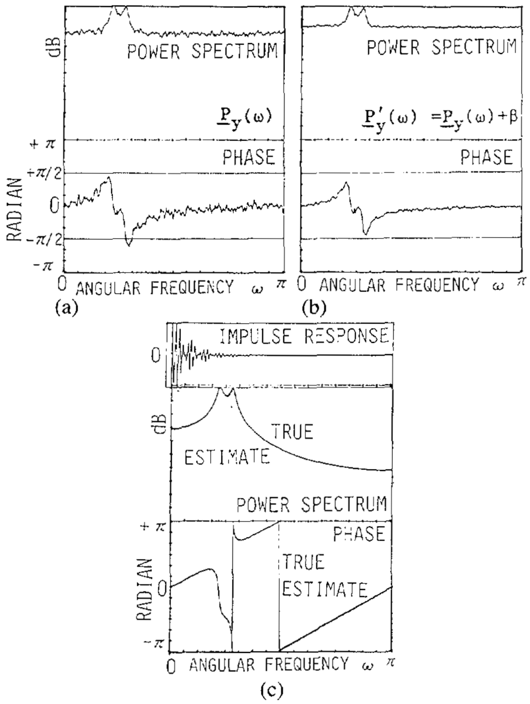


Fig. 5. In case of high SNR, by adding the positive constant to the power spectrum, the accurate AR parameters are estimated from the proposed method (for the first example). (a) In case of SNR = +5 dB, the evaluation of the phase error $\Delta\Theta(\omega)$ of (28) is unstable since $|\Theta_{\min}(\omega)|$ of the minimum phase signal obtained from the power spectrum estimate $P_y(\omega)$ is larger than $\pi/2$ for the frequencies near the central frequencies of the poles. (b) By adding the positive constant β to the power spectrum $P_y(\omega)$, the absolute value of $\Theta_{\min}(\omega)$ calculated from $P'_y(\omega) = P_y(\omega) + \beta$ is less than $\pi/2$ for all ω. (c) A satisfactory AR model is estimated by the proposed method.

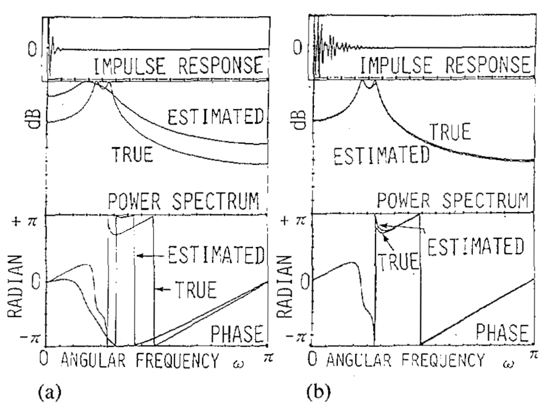


Fig. 6. The estimated power spectrum and phase obtained by the two methods (for the first example) (SNR = −5 dB). (a) OMYW method. (b) The proposed method.

AR-plus-noise process (SNR = +4 dB). Fig. 8 shows the estimated spectrum of the AR process obtained by the above two methods. The SNR is about +4 dB. The AR model estimated by the OMYW method has a large degree of error as shown in Fig. 8(a). However, by using the proposed method, a satisfactory spectrum is obtained.

VI. Experiment and the Results

We applied the proposed method to the estimation of characteristics of the resonant vibration caused by many flaws on the rough race of a small-sized ball bearing.

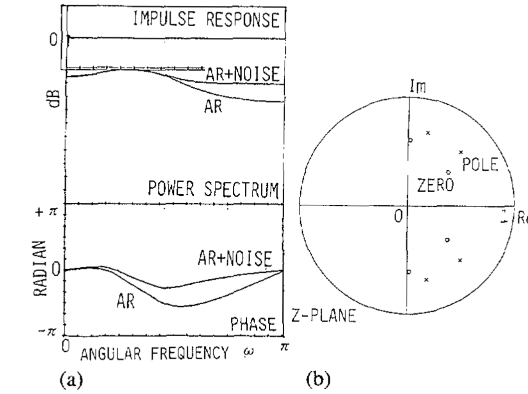


Fig. 7. The characteristics of the AR model (for the second example). (a) The fourth-order AR model of (33) and the AR model buried in the white noise (SNR = +4 dB). (b) The poles of the AR model and the spectral zeros introduced due to the additive noise.

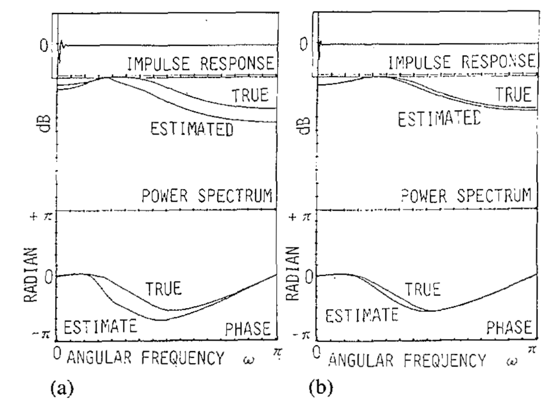


Fig. 8. The estimated power spectrum and phase obtained by the two methods (for the second example (SNR = +4 dB). (a) OMYW method. (b) The proposed method.

The resonant vibration signal is detected by the following procedure [15], [16]. The outer ring is fixed by imposing an axial pressure, and the inner one revolves at a constant speed of 1800 RPM. Under such conditions, a flaw on the race causes radial movement of the outer ring, and the resulting signal from the movement is sensed by a vibration pickup attached to the outer ring in an Anderson meter [15]. The signal is amplified and filtered through a high-pass filter to eliminate the primary frequency component (= 30 Hz) corresponding to the rotation of the inner ring. The filtered signal is A/D converted with a 12 bit A/D converter at a sampling period of 30 μs.

Since the resonant vibration signal detected above is buried in the high level noise, the detected signal $z(n)$ is approximately expressed as the AR-plus-noise process in (1) and (2). To estimate the characteristics of the resonant vibration $x(n)$ using $z(n)$, the following five processes are carried out (see Fig. 9).

1) After multiplying each input sequence of 83 normal bearings having smooth races by the Hanning window, 2048-point FFT's are carried out. The power spectra are averaged for the 32 successive nonoverlapping sections. Then, by summing up the power spectra of 83 normal bearings, the averaged power spectrum $P_s(\omega)$ is obtained. Fig. 9(a) shows the inverse of $P_s(\omega)$. If the ad-

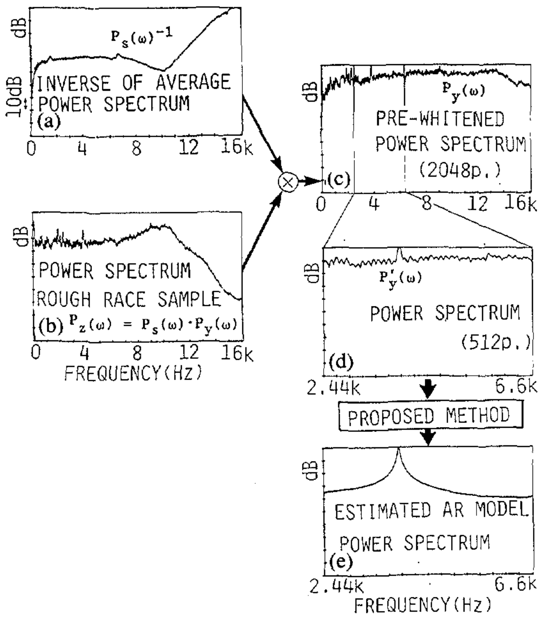


Fig. 9. The estimated AR spectrum of the resonant vibration of a ball bearing obtained by the proposed method. (a) The inverse spectrum $P_s(\omega)^{-1}$ of the average power spectrum of the vibration signals of the 83 normal ball bearings. (b) The power spectrum $P_z(\omega)$ of the vibration signal of a defective ball bearing with rough races. (c) The prewhitened power spectrum $P_y(\omega)$ obtained by multiplying the power spectrum $P_z(\omega)$ in (b) by $P_s(\omega)^{-1}$ of (a). (d) The power spectrum $P'_y(\omega)$ obtained by selecting and splitting the important frequency band of the power spectrum $P_y(\omega)$ of (c). (e) Estimated second-order AR model obtained using the proposed method from $P'_y(\omega)$ of (d).

ditive noise is white, $P_s(\omega)$ denotes the power spectrum of the transfer function of the signal measurement process.

2) After multiplying the sequence of a defective bearing having a rough race by the Hanning window, 2048-point FFT's are carried out. The power spectra are summed up for the 32 nonoverlapping successive sections. The resulting averaged power spectrum $P_z(\omega)$ denotes the product of $P_s(\omega)$ and the power spectrum $P_y(\omega)$ of the AR-plus-noise signal.

3) By dividing the power spectrum $P_z(\omega)$ by $P_s(\omega)$, prewhitened power spectrum $P_y(\omega)$ is obtained as shown in Fig. 9(c).

4) We selected the important frequency band of $P_y(\omega)$ and clipped it from $P_y(\omega)$, and then the power spectrum $P'_y(\omega)$ is obtained as shown in Fig. 9(d). The number of points of the power spectrum $P_y(\omega)$ is 512.

5) It is difficult for the proposed method to estimate the AR parameters when the order of the AR model is not known *a priori*. However, it is known that the central frequency of the resonant vibration of the outer ring in ball bearing is about 4 kHz [15], [16]. Thus, we applied our method to estimate the poles of the real vibration under the assumption that the vibration is described by the second-order AR model in the frequency band around 4 kHz. Using the proposed method, the AR parameters of the vibration signal are obtained from $P'_y(\omega)$. Fig. 9(e) shows the characteristics of the estimated resonant vibration. The initial value of the Marquardt interation used in the proposed method is set to be the unit impulse as follows:

$$x_{\text{zero}}(n:0) = \begin{cases} 1 & (n = 0) \\ 0 & (n \neq 0). \end{cases} \tag{34}$$

Using the initial value, the denominator of the right-hand side in (28) is always positive for all frequencies. Then, as described in Section IV-C, by confining the denominator to a positive value in each iteration, 12 iterations were needed to achieve convergence for this example. Since the input of the proposed method is the power spectrum of the observed signal, the prewhitening process is simple as described above.

VII. Concluding Remarks

A new spectrum estimate method which is based on phase matching is proposed for the accurate estimation of the parameters of the AR process buried in high level noise. From experiments with numerical examples, the AR parameters of the AR-plus-noise process at low SNR are estimated accurately using our new method.

Two issues remain for future research as follows.

1) The poles of the AR model should be derived by computing the roots of the polynomial obtained from the finite length all-zero model estimate as described in Section II.

2) In the experiments, the order of the AR model is already known. However, for an unknown signal, it is difficult to choose the order of the AR model. In such a case, the ordinary spectrum estimation methods use computationally efficient order recursive techniques to find AR parameters for the various orders of the model. However, since the method proposed in this paper uses nonlinear optimization for phase matching, it is difficult to use such recursive techniques.

These important issues are currently under investigation.

References

[1] S. M. Kay and S. L. Marple, Jr., "Spectral analysis—A modern perspective," *Proc. IEEE*, vol. 69, pp. 1380–1419, Nov. 1981.

[2] J. A. Cadzow, "Spectral estimation: An over-determined rational model equation approach," *Proc. IEEE*, vol. 70, pp. 907–939, Sept. 1982.

[3] S. M. Kay, "The effects of noise on the autoregressive spectral estimator," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-27, pp. 478–485, Oct. 1979.

[4] ——, "Noise compensation for autoregressive spectral estimators," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 292–303, June 1980.

[5] B. Friedlander and B. Porat, "A spectral matching technique for ARMA parameter estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 338–343, Apr. 1984.

[6] A. V. Oppenheim *et al.*, *Digital Signal Processing.* Englewood Cliffs, NJ: Prentice-Hall, 1975.

[7] T. F. Quatieri, Jr. and A. V. Oppenheim, "Iterative techniques for minimum phase signal reconstruction from phase or magnitude," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 1187–1193, Oct. 1981.

[8] B. Yegnanarayana and D. K. Saika, "Significance of group delay functions in signal reconstruction from spectral magnitude or phase," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 610–622, June 1984.

[9] J. D. Markel and A. H. Gray, Jr., *Linear Prediction of Speech.* New York: Springer-Verlag, 1976.

[10] M. H. Hayes, J. S. Lim, and A. V. Oppenheim, "Signal reconstruction from phase or magnitude," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 672–680, Dec. 1980.

[11] G. A. Merchant and T. W. Parks, "Reconstruction of signals from phase: Efficient algorithms, segmentation, and generalizations," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, pp. 1135–1147, Oct. 1983.

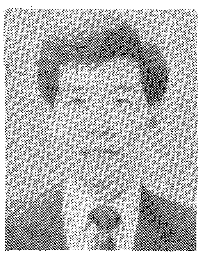
[12] S. L. S. Jacoby, J. S. Kowalik, and J. T. Pizzo, *Iterative Methods for Nonlinear Optimization Problems.* Englewood Cliffs, NJ: Prentice-Hall, 1972.

[13] T. J. Ulrych and T. N. Bishop, "Maximum entropy spectral analysis and autoregressive decomposition," *Rev. Geophys. Space Phys.*, vol. 13, pp. 183-200, Feb. 1975.

[14] C. Y. Espy and J. S. Lim, "Effects of additive noise on signal reconstruction from Fourier transform phase," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, pp. 894–898, Aug. 1983.

[15] H. Kanai, M. Abe, and K. Kido, "Detection and discrimination of flaws in ball bearings by vibration analysis," *J. Acoust. Soc. Japan*, vol. 7, pp. 121–131, Mar. 1986.

[16] ——, "Detection of slight defects in ball bearings by nonperiodic analysis," *J. Acoust. Soc. Japan*, vol. 7, pp. 219–228, July 1986.



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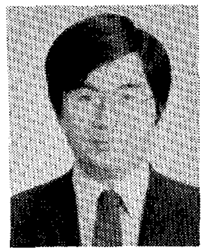

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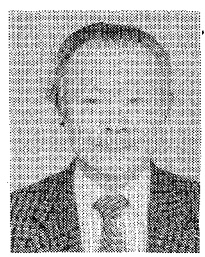

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